Two-dimensional Modal Logics with Difference Relations

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TWO-DIMENSIONAL MODAL LOGICS WITH DIFFERENCE RELATIONS

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Abstract

In this thesis, we explore various computational and axiomatisation problems relating to two-dimensional modal logics that exhibit some modest capacity to count. In particular, we consider modal products in which at least one component is the logic of difference (inequality) relations. These formalisms are connected with finite variable fragments of first-order logic and first-order modal logics, extended with some additional counting quantifiers. The contributions provided herein serve as a case study to better steer investigation into more general principles governing the interactions between modal logics, and into understanding the interactions between first-order quantifiers.

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Chapter 1

Introduction

Modal logic has a rich history originating in the classical considerations of philosophers such as Aristotle. Originally conceived as the logic of necessary and contingent truths (so-called *alethic logic*), further study has shown the same underlying semantics to encapsulate a vast range of other natural linguistic phenomena; such disparate phenomena as *spatial* and *temporal* reasoning, *epistemic* and *doxastic* reasoning, relating to one's knowledge and belief, and *deontic* reasoning of one's obligations and permissions [14, 47].

While there are many applications of modal logics working in isolation, it is often their interactions in which we are most interested. The notion of *products* of propositional modal logics, introduced by Segerberg [113] and Shehtman [115], provides a conceptually appealing approach to combining modal logics. Products of modal logics are connected to several other 'many-dimensional' logical formalisms, such as finite variable fragments of classical, intuitionistic, modal, and temporal first-order logic, as well as temporal epistemic logics [32], and extensions of description logics with 'dynamic' features [9, 11, 135, 136, 137]. The product construction has been studied extensively since its inception [38, 73], and has borne many applications in computer science and artificial intelligence [100, 12, 35].

Unlike for unimodal logic, and for logics having multiple non-interacting modal operators — known as *fusions* — there is a scarcity of general results for logics with interacting modal operators, and for products of modal logics, in particular. However, there are some detailed 'case studies' concerning products of some well-known 'standard' modal systems. For example, two-dimensional products of the form $L \times \mathbf{S5}$ are typically decidable, where $\mathbf{S5}$ is the logic of all equivalence relations introduced by Lewis [79]. Furthermore, $L \times \mathbf{S5}$

is known to be finitely axiomatisable, whenever L is Horn-axiomatisable. Far less is known about products involving von Wright's 'logic of elsewhere'. Von Wright's logic, denoted by **Diff**, is characterised by the class of all difference relations, in which every possible world is accessible from every other *distinct* possible world. This is the underlying logic of the *difference operator* [26, 27], and is connected to *nominals* [44] and *graded modalities* [8], all of which find many applications in description logics [10]. It is also a simple example of a non-Horn axiomatisable modal logic. The product logic **Diff** $\times$ **Diff** is related to the two-variable, substitution and equality-free fragment of first-order logic with counting quantifiers $\exists_{>m}x, \exists_{>m}y$, for $m = 0, 1$.

Despite the similarities that **Diff** shares with **S5**, in the structure of their frames, their computational complexity, and their axiomatisations, their respective interactions in two dimensions differ considerably; as this thesis shall demonstrate. The contributions provided herein serve as a case study to better steer investigation into more general principles governing the interactions between modal logics.

For multimodal logics L, contained herein, we will be chiefly concerned with the following types of problems:

- The *axiomatisation problem*: Is it possible to characterise the theorems of L by means of some explicit (recursive or even finite) set of axioms, closed under the usual deduction rules of Modus Ponens, Necessitation and Uniform Substitution? A necessary condition for axiomatisability is that the theorems of L are recursively enumerable.

- The *decision problem*: Is a given formula φ valid in every L-model; that is to say, is φ a theorem of L? Dually, we have the *satisfiability problem*: Is a given formula φ satisfiable is some L-model? We will be interested in the computational (worst case) complexity of the decision / satisfiability problem.

- The *finite frame problem*: Is a given finite frame $\mathfrak{F}$ a frame for L? If L is finitely axiomatisable, then this problem is trivially decidable, however this need not be true in general. This problem can often be reduced to the decision problem, but can, in many cases, be genuinely simpler.

- The *finite model property*: Is every non-theorem of L refutable in some finite frame for L? An affirmative answer to this question provides us with an effective procedure by which we can enumerate all non-theorems of L, whenever the finite frames problem for L is decidable. This is the case when L is finitely axiomatisable, in which case the theorems of L are also recursively enumerable. Hence, every finitely axiomatisable logic with the finite model property is decidable.

Thesis Structure and Summary of Results

The thesis is divided into three parts, together with an appendix: Part I, comprising Chapters 2–4, serves as a general introduction and provides the necessary background and definitions for the topics contained herein. In Part II, we consider problems relating to the axiomatisation and computational properties of various two-dimensional modal product logics involving the difference relation. In particular:

Chapter 5

In this chapter we tackle problems relating to the axiomatisation and finite frame problems of logics of the form $L \times \mathbf{Diff}$. In Section 5.1, we show that no logic between $\mathbf{K} \times \mathbf{wK5}$ and $\mathbf{S5} \times \mathbf{Diff}$ can be axiomatised using only finitely many propositional variables, where $\mathbf{wK5}$ denotes a sublogic of $\mathbf{Diff}$, characterised by the class of all of *weakly-Euclidean* relations. In Section 5.2 we show that the logic characterised by only *square* product frames for $\mathbf{Diff} \times \mathbf{Diff}$ cannot be finitely axiomatised over $\mathbf{Diff} \times \mathbf{Diff}$; itself not finitely axiomatisable.

Product logics always validate the *commutativity* and *Church–Rosser* axioms, describing the interactions between the two modalities. Hence, in considering possible axiomatisations for $L_1 \times L_2$, it is often a good starting point to consider the *commutator* $[L_1, L_2]$: the smallest bimodal logic containing both L_1, L_2 and the aforementioned interactions. In Section 5.3, we discuss products falling outside the aforementioned interval of logics and show that $\mathbf{Alt} \times L = [\mathbf{Alt}, L]$, whenever L is any *canonical* logic, and where $\mathbf{Alt}$ is the logic of all functional relations. In particular, $\mathbf{Alt} \times \mathbf{Diff} = [\mathbf{Alt}, \mathbf{Diff}]$ and $\mathbf{Alt} \times \mathbf{K4.3} = [\mathbf{Alt}, \mathbf{K4.3}]$, and consequently, both $\mathbf{Alt} \times \mathbf{Diff}$ and $\mathbf{Alt} \times \mathbf{K4.3}$ are finitely axiomatisable. Here $\mathbf{K4.3}$ denotes the (non-Horn axiomatisable) logic of all linear orders. These are the first examples of products that coincide with the respective commutators, whose components are not both Horn-axiomatisable.

In Section 5.4, we provide a full classification of the finite frames for $\mathbf{S5} \times \mathbf{Diff}$, thereby providing a polynomial time algorithm for its finite frame problem, despite the lack of any known finite axiomatisation. In Section 5.5 we discuss how this classification theorem may be generalised to describe the finite frames for **Diff** $\times$ **Diff**.

Chapter 6

In this chapter we discuss various versions of the finite model property of bimodal logics related to products of the form $L \times$ **Diff**. In Section 6.1 we show that no normal extension of [**wK5**, **wK5**], having $(\omega, \neq) \times (\omega, \neq)$ among its frames, can be characterised by its finite frames. In particular, it follows that **Diff** $\times$ **Diff** lacks the finite model property. Note that while the product logic **Diff** $\times$ **Diff** is characterised by the class of all products of difference frames, there may be many more non-product frames that validate all the axioms of **Diff** $\times$ **Diff**. Hence, this result is more general than saying that **Diff** $\times$ **Diff** is not characterised by its finite product frames — a result that follows easily from the lack of finite model property of two-variable first-order logic with counting quantifiers.

On the other hand, in Section 6.2 we show by a variation on the well-known quasimodel technique that **S5** $\times$ **Diff** does have the finite model property, even with respect to its product frames. However, this is a fine line: If we restrict our attention only to square product frames, in which both component frames have the same cardinality, then we find that **S5** $\times$ **Diff** cannot be characterised without invoking square frames of infinite cardinality.

Chapter 7

In this chapter we discuss some decision problems of commutators involving **Diff**. The current literature provides no techniques with which to handle the decision problems of commutators that do not coincide with their respective products, and whose components are not both Horn-axiomatisable. In particular, the decidability and complexity of the decision problems for both [**Diff**, **Diff**] and [**S5**, **Diff**] cannot be ascertained by any existing techniques — by the results of Section 5.1, neither of these logics coincide with the respective products.

In Section 7.1, we introduce a novel approach to obtaining decidability results for the decision problems for each of these logics, together with elementary upper bounds on their respective complexities. We achieve these results with the aid of a recursive satisfiability-preserving translation of each commutator to their corresponding product logic. In Section 7.2, we employ a variation on this technique to show that [**S5**, **Diff**] is characterised by its finite frames.

Chapter 8

In this chapter we consider lower bounds on the complexity of the decision problem for various products of the form $L \times \mathbf{Diff}$. Product frames are always 'grid-like' by definition. Hence, in those cases where coordinate-wise 'universal' and 'next-time' operators are both available, it becomes straightforward to obtain lower bounds by using reductions from various complex grid-based problems, such as tiling problems or Turing machine problems.

For example, it is easy to see that the decision problem for $\mathbf{K}_u \times \mathbf{K}_u$ is undecidable, where $\mathbf{K}_u$ is the logic $\mathbf{K}$ of all frames enriched with the *universal modality* [121]. Previous lower bound proofs [84, 102, 41] on product logics over *transitive* frames overcome the lack of next-time operators by 'diagonally' encoding the $\omega \times \omega$-grid in product models. Here we develop a novel technique, making direct use of the grid-like structure of product frames, to obtain undecidability results for a host of product logics using reductions from various (Minsky) counter machine problems. The results of this chapter are in sharp contrast with the corresponding results for logics of the form $L \times \mathbf{S5}$, whose decision problems are all known to be decidable.

In particular, in Section 8.2 we introduce the technique for cases when a 'horizontal' next-time operator is still available. We show that $\mathbf{K}_u \times \mathbf{Diff}$ is undecidable (but recursively enumerable), while the product of $\mathbf{Diff}$ and $\mathbf{K}$ enriched with the *common knowledge* (*transitive closure*) operator is not even analytic. The same is true of $\mathbf{PTL}_{\bigcirc\Box} \times \mathbf{Diff}$, in which $\mathbf{PTL}_{\bigcirc\Box}$ denotes the bimodal *propositional temporal logic* characterised by the frame $(\omega, S, <)$, where S is the successor relation on ω. The results of this section are published in [58].

In Section 8.3 we sharpen this technique and discuss products of the form $L \times \mathbf{Diff}$, where L is characterised by some class $\mathcal{C}$ of *linear orders*, without next-time. We prove that the decision problem is undecidable whenever $\mathcal{C}$ comprises any class of linear orders containing $(\omega, <)$, and highly undecidable whenever $\mathcal{C}$ comprises any class of modally discrete linear orders containing $(\omega, <)$, or any class of Noetherian linear orders containing $(\omega + 1, >)$. In particular, $\mathbf{K4.3} \times \mathbf{Diff}$ is undecidable, while $\mathsf{Log}(\omega, <) \times \mathbf{Diff}$ is highly undecidable. Note that it remains open whether the technique can be extended to cases of 'branching', transitive frames (without next-time). In particular, it is unknown whether the product of $\mathbf{Diff}$ and the logic $\mathbf{K4}$ of all transitive frames is decidable.

Finally, in Section 8.4 we show that the results of Section 8.3 are genuine generalisations of the undecidability results obtained in [84, 102], giving a polynomial reduction from the decision problem for $L \times \mathbf{Diff}$ to that of $L \times \mathbf{K4.3}$, whenever L is Kripke complete.

Furthermore, despite the shared coNP-completeness of the decision problems for both **K4.3** and **Diff**, one cannot hope for a general reverse reduction, since **Diff** $\times$ **Diff** is decidable, while **Diff** $\times$ **K4.3** is undecidable. The results of Sections 8.3 and 8.4 are published in [59].

Part III goes beyond the regular product construction to consider various *relativisations* of products, and products equipped with a *diagonal element*. In particular:

Chapter 9

In this chapter we explore a variation on the standard product construction, motivated by connections between modal product logics of the form $L \times \mathbf{S5}$ and $L \times \mathbf{Diff}$, and various one-variable fragments of first-order modal logics. Owing to philosophical debate as to how we should interpret statements involving both modal operators and first-order quantifiers, investigation into such first-order modal logics has motivated a range of possible semantics, including those in which the domain of interpretation is permitted to either *expand* or *contract*, relative to the direction of modal accessibility relation. This motivates the consideration of *relativised product logics*, characterised by subframes of product frames that, similarly, expand or contract with respect to a given dimension.

It is easy to see that the decision problems of both expanding and contracting relativised products are reducible to that of the standard products. Here, we investigate whether they can be genuinely simpler.

In Section 9.4, we consider relativised products with contracting domains and show their decision problems to be often as complex as their non-relativised counterparts. In particular, we show that over *decreasing* domain models, the decision problem for $\mathbf{K4.3} \times^{dec} \mathbf{Diff}$ is undecidable, while that of $\mathsf{Log}((\omega, <) \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$ is highly undecidable.

In Section 9.5, we consider relativised products with expanding domains. We employ the techniques described in Section 8.3, with the aid of *unreliable* counter machine problems, to obtain lower complexity bounds for various expanding product logics. In particular, we show that over *expanding* domain models, the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is non-elementary, whenever $\mathcal{C}$ comprises any class of strict linear orders containing $(\omega, <)$ and is even non-primitive recursive, when $\mathcal{C}$ comprises the class of all finite linear orders. Furthermore, we show that the decision problem for $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is undecidable.

These lower bounds are notably weaker than those respective lower bounds obtained in

Section 8.3, owing to the lower complexity of unreliable counter machines compared with their reliable counterparts. However, we demonstrate several cases where these bounds are optimal, via model-level reductions to known results.

This chapter builds upon results published in [59] with some additional unpublished results. The proofs presented in this chapter differ slightly from those appearing in [59], in their use of *incrementing* counter machine problems rather than the *lossy* counter machine problems originally employed.

Chapter 10

A variation on the standard translation identifies the product logic $\mathbf{S5} \times \mathbf{S5}$ with the two-variable, equality-free, (substitution-free) fragment of first-order logic. In this modal setting, equality can be modelled with an additional 'diagonal' constant, interpreted in square frames as the identity relation. In this chapter we consider the decision problems of products of *arbitrary* modal logics expanded with this additional feature. We show that, unlike the two-variable fragment whose validity problem remains coNExpTime-complete, both with and without equality, the addition of an additional dimension-joining diagonal operator can often lead to a considerable increasing in the complexity of product logics.

In Section 10.2, we first establish a connection between delta products and regular products. In particular, we show that the global consequence problem for certain product logics can be reduced to the decision problem for their respective delta products. This provides us with undecidable lower bounds for several delta product logics whose delta-free counterparts are decidable. In particular, we show that the decision problems for the delta products $\mathbf{K} \times^{\delta} \mathbf{K}$ and $\mathbf{K} \times^{\delta} \mathbf{K4}$ are undecidable.

However, there are limitations to this approach. Indeed there are cases where such a reduction is either unhelpful or demonstrably non-existent. In particular, we cannot infer the undecidability of the decision problem for $\mathbf{K} \times^{\delta} \mathbf{S5}$, as the global consequence problem for $\mathbf{K} \times \mathbf{S5}$ is known to be decidable.

In Section 10.3, we introduce the notion of computation by means of *faulty approximations* as a novel variation on unreliable counter machines. Unlike lossy and incrementing counter machines, we show that computation by faulty approximation is *Turing-complete*. In Section 10.4.1, we exploit the greater flexibility of this new formalism to obtain undecidable lower bounds for a host of delta products, using a variation on the techniques described in Chapter 8. Among which, we show that the decision problem for $\mathbf{K} \times^{\delta} \mathbf{S5}$ is undecidable, despite the decidability of both the decision problem and the global consequence problem for $\mathbf{K} \times \mathbf{S5}$.

In Section 10.4.2, we extend this technique to delta products in which the first component is characterised by some class of linear orders. In particular, we show that the decision problems for $\mathbf{K4.3} \times^{\delta} \mathbf{K}$ and $\mathbf{K4.3} \times^{\delta} \mathbf{S5}$ are both undecidable. Notably, the complexity of the global consequence problem for $\mathbf{K4.3} \times \mathbf{K}$ remains open, while that of $\mathbf{K4.3} \times \mathbf{S5}$ is known to be decidable.

Finally, in Section 10.5, we probe the limitation of this approach and show that the delta product $\mathbf{K} \times^{\delta} \mathbf{Alt}$ — lying outside the remit of the aforementioned results — has the exponential product fmp, as defined in Section 2.1.2, and is thus decidable. The results of this chapter are to be published in [57].

Appendices

In Appendix A, we include for reference a brief overview of some definitions and notation from complexity theory and recursion theory that will be used throughout this thesis. Appendix B contains some additional results pertaining to the ω-Reachability problem for incrementing counter machines, discussed in Section 9.3. They are included here for the sake of completeness.

A word on notation

Throughout this thesis we will treat natural numbers $0, 1, 2, \ldots$ as ordinals, adopting the convention that $0 := \emptyset$ is the empty set, and $(n + 1) := n \cup \{n\} = \{0, 1, \ldots, n\}$, for each $n > 0$. We denote by $\omega := \{0, 1, 2, \ldots\}$ the first infinite ordinal, comprising the set of all natural numbers, and take $(\omega + 1) := \omega \cup \{\omega\}$ to be its immediate successor — ordinals greater than $(\omega + 1)$ will not be considered in this thesis. We write $\alpha < \beta$ to mean that $\alpha \in \beta$, and $\alpha \leq \beta$ to mean that $\alpha < \beta$ or $\alpha = \beta$, for all ordinals α and β.

We assume the standard notation for the set of all integers $\mathbb{Z}$, the set of all rational numbers $\mathbb{Q}$, and the set of all real numbers $\mathbb{R}$.

For sets X and Y, we denote by Y^X the set of all functions $f : X \to Y$, with domain X and co-domain Y. In particular, we take 2^X to be the set of all functions $f_A : X \to \{0, 1\}$, each of which we may associate with some subset $A = \{x \in X : f_A(x) = 1\} \subseteq X$. Hence we may associate 2^X with the *powerset* of X, comprising all subsets of X.

Part I

Background

Chapter 2

Basic Modal Logic

2.1 Syntax and Semantics

Throughout this thesis, we will primarily be considering the following *n-modal language*, comprising a countably infinite set of propositional variables $\textsc{prop} = \{p_0, p_1, p_2, \dots\}$, together with the following logical symbols for *negation* $\neg$ and *conjunction* $\wedge$, as well as a set of (unary†) *modal operators* (or *modalities*) $\Diamond_1, \dots, \Diamond_n$. The set of all *n-modal formulas* $\mathcal{ML}_n$ collects all those strings defined in accordance to the following grammar:

$$\varphi \ ::= \quad p_j \ | \ \neg\varphi \ | \ (\varphi_1 \wedge \varphi_2) \ | \ \Diamond_1\varphi \ | \ \dots \ | \ \Diamond_n\varphi$$

where $p_j \in \textsc{prop}$. A *subformula* of φ is any substring of φ that is itself a formula of $\mathcal{ML}_n$. We denote by $\mathrm{sub}(\varphi)$, the set of all subformulas of φ. We take the *size* of φ to be the cardinality of $\mathrm{sub}(\varphi)$. The *modal depth* of φ, describing the maximal nesting of modal operators, is defined inductively as follows:

$$\mathrm{md}(p_j) = 0 \quad \text{for } p_j \in \textsc{prop}, \qquad \mathrm{md}(\neg\varphi) = \mathrm{md}(\varphi)$$
$$\mathrm{md}(\varphi_1 \wedge \varphi_2) = \max\big(\mathrm{md}(\varphi_1), \mathrm{md}(\varphi_2)\big), \qquad \mathrm{md}(\Diamond_i\varphi) = \mathrm{md}(\varphi) + 1,$$

for $1 \leq i \leq n$. In the interest of readability, we adopt the standard abbreviations for *vērum* $\top$, *falsum* $\bot$, *disjunction* $\vee$, *implication* $\rightarrow$, and *equivalence* $\leftrightarrow$, as outlined in Table 2.1, below.

†Cases of modal operators of arity ≥ 2 will not be considered in this thesis, while *nullary* modal operators, taking no arguments, will be introduced later in Chapter 10.

In addition to this, for each $k < \omega$, we define $\Diamond_i^k\varphi$ inductively, by taking $\Diamond_i^0 := \varphi$ and $\Diamond_i^{k+1}\varphi := \Diamond_i\Diamond_i^k\varphi$, for all $1 \le i \le n$. Furthermore, we define $\Box_i^{\le m}\varphi := \bigwedge_{k=0}^m \Box_i^k\varphi$, for all $m < \omega$, where $\Box_i^k\varphi := \neg\Diamond_i^k\neg\varphi$. Given a sequence of formulas $\varphi_0, \ldots \varphi_k$, we denote their collective conjunction and disjunction, by taking

$$\bigwedge_{i=0}^{k} \varphi_i := (...(\varphi_0 \wedge \varphi_1) \ldots \wedge \varphi_k) \qquad \text{and} \qquad \bigvee_{i=0}^{k} \varphi_i := (...(\varphi_0 \vee \varphi_1) \ldots \vee \varphi_k).$$

$\bot$	$:=$	$p_0 \wedge \neg p_0$	$\top$	$:=$	$p_0 \vee \neg p_0$
$(\varphi_1 \vee \varphi_2)$	$:=$	$\neg(\neg\varphi_1 \wedge \neg\varphi_2)$	$(\varphi_1 \to \varphi_2)$	$:=$	$\neg\varphi_1 \vee \varphi_2$
$(\varphi_1 \leftrightarrow \varphi_2)$	$:=$	$(\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$	$\Box_i\varphi$	$:=$	$\neg\Diamond_i\neg\varphi$
$\Diamond_i^+\varphi$	$:=$	$\varphi \vee \Diamond_i\varphi$	$\Box_i^+\varphi$	$:=$	$\varphi \wedge \Box_i\varphi$

Table 2.1: Common abbreviations in $\mathcal{ML}_n$.

2.1.1 Modal Logics

A *(normal) n-modal logic* is defined to be any set of formulas $L \subseteq \mathcal{ML}_n$ containing all propositional tautologies[†] together with the formulas

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q), \qquad \text{for all } 1 \le i \le n,$$

that is closed under the deduction rules: *modus ponens* **(MP)**, *necessitation* **(Nec)** and *uniform propositional substitutions* **(Subst)**.

(MP)	if $\varphi \in L$ and $\varphi \to \psi \in L$ then $\psi \in L$,
(Nec)	if $\varphi \in L$ then $\Box_i\varphi \in L$ for all $1 \le i \le n$,
(Subst)	if $\varphi \in L$ then $\varphi\sigma \in L$ where σ is a uniform substitution, replacing propositional variables with $\mathcal{ML}$-formulas.

Table 2.2: Deduction rules for n-modal logics.

[†]It is sufficient to demand that L include the following axioms of propositional calculus: (i) $p \to (q \to p)$, (ii) $\big(p \to (q \to r)\big) \to \big((p \to q) \to (p \to r)\big)$, (iii)$\neg\neg p \to p$ [23].

We say that a formula $\varphi \in \mathcal{ML}_n$ is a *theorem* of L if $\varphi \in L$. With each logic $L \subseteq \mathcal{ML}_n$, we associate its *decision problem*, which asks whether there is an effective procedure for distinguishing those theorems of L from those non-theorems of L. We say that the decision problem for L is *decidable* should such a procedure exist. We describe the computational complexity of such modal logics in terms of their position in the complexity hierarchy, detailed in Appendix A. Where no such procedure exists, we say that the decision problem for L is *undecidable* and characterise its complexity according to the arithmetic and analytic hierarchies. The existence of such undecidable logics is an immediate consequence of there being uncountably many modal logics, yet only countably many possible decision procedures.

A logic L is said to be *recursively enumerable (r.e.)* if there is some decision procedure, whereby we may enumerate all theorems of L, and *co-recursively enumerable (co-r.e.)* if there is some procedure whereby we may enumerate all non-theorems of L. Clearly any logic that is both recursively enumerable and co-recursively enumerable is decidable.

$\mathbf{T}$	$:=$	$\mathbf{K} + (T)$	$=$	$\mathbf{K}$	$+$	$\Box p \to p$
$\mathbf{Alt}$	$:=$	$\mathbf{K} + (alt)$	$=$	$\mathbf{K}$	$+$	$\Diamond p \to \Box p$
$\mathbf{K4}$	$:=$	$\mathbf{K} + (4)$	$=$	$\mathbf{K}$	$+$	$\Box p \to \Box\Box p$
$\mathbf{S5}$	$:=$	$\mathbf{T} + (5)$	$=$	$\mathbf{T}$	$+$	$\Diamond p \to \Box\Diamond p$
$\mathbf{K4.3}$	$:=$	$\mathbf{K4} + (.3)$	$=$	$\mathbf{K4}$	$+$	$\Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$
$\mathbf{wK4}$	$:=$	$\mathbf{K} + (w4)$	$=$	$\mathbf{K}$	$+$	$\Diamond\Diamond p \to (p \vee \Diamond p)$
$\mathbf{wK5}$	$:=$	$\mathbf{K} + (w5)$	$=$	$\mathbf{K}$	$+$	$\Diamond p \to \Box(p \vee \Diamond p)$
$\mathbf{Diff}$	$:=$	$\mathbf{wK5} + (B)$	$=$	$\mathbf{wK5}$	$+$	$p \to \Box\Diamond p$
$\mathbf{GL}$	$:=$	$\mathbf{K4} + (l\ddot{o}b)$	$=$	$\mathbf{K4}$	$+$	$\Box(\Box p \to p) \to \Box p$
$\mathbf{Grz}$	$:=$	$\mathbf{S4} + (grz)$	$=$	$\mathbf{S4}$	$+$	$\Box\big(\Box(p \to \Box p) \to p\big) \to \Box p$

Table 2.3: Some common unimodal logics.

Given two n-modal logics L_1 and L_2, we say that L_2 is *(finitely) axiomatisable* over L_1 if there is some (finite) set of formulas $\Gamma \subseteq \mathcal{ML}_n$ such that L_2 is the smallest modal logic subsuming $L_1 \cup \Gamma$; in such cases we write that $L_2 = L_1 + \Gamma$. The formulas of Γ are appropriately referred to as *axioms* for L_2 over L_1. Furthermore, we say that a logic L is *finitely axiomatisable* if it is finitely axiomatisable over the minimal n-modal logic $\mathbf{K}_n$.

Some of the more common unimodal logics, together with their traditional names, are given in Table 2.3. In addition to those given in Table 2.3, are the logics $\mathbf{S4} := \mathbf{K4} + (T)$ and $\mathbf{S4.3} := \mathbf{K4.3} + (T)$, axiomatised with the addition of the axiom $(T) := \Box p \to p$, and the logics $\mathbf{GL.3} := \mathbf{GL} + (.3)$ and $\mathbf{Grz.3} := \mathbf{Grz} + (.3)$, axiomatised with the addition of the axiom $(.3) := \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$.

The set of all modal logics form a lattice under set-inclusion, with $\mathbf{K}_n$ denoting the minimal n-modal logic and $\mathsf{Log}\ \emptyset$ denoting the maximal, *inconsistent logic*, comprising all n-modal formulas. We say that L' is a *normal extension* of L, whenever L' is a normal modal logic subsuming L. The lattice structure for those unimodal logics described above is illustrated in Figure 2.1[†]

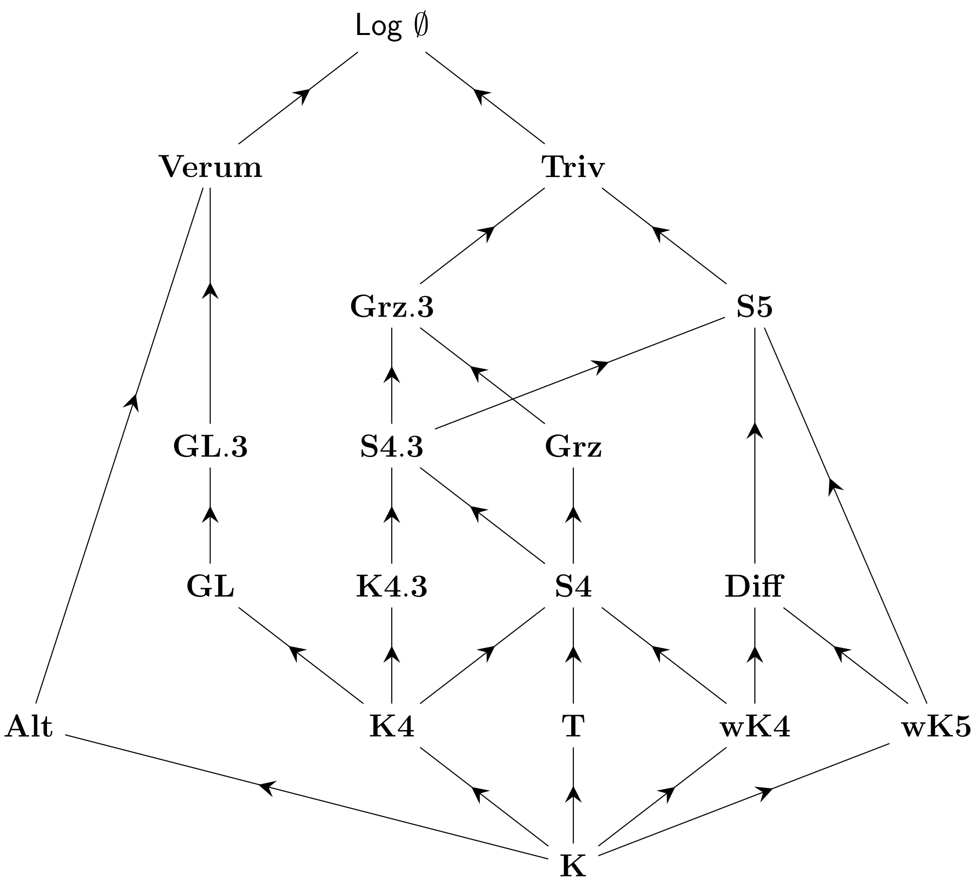


Figure 2.1: Lattice of 'standard' unimodal logics.

[†]Here, $\mathbf{Verum} := \mathbf{K4} + \Box p$ and $\mathbf{Triv} := \mathbf{K4} + (\Box p \leftrightarrow p)$ denote the (only) two maximal *consistent* modal logics [82]. These logics will not be discussed any further, herein.

2.1.2 Frames

Modal formulas are interpreted over *Kripke frames* $\mathfrak{F} = (W, R_1, \dots, R_n)$, where W is a non-empty set of *possible worlds* and each $R_i \subseteq W \times W$ describes a binary *accessibility relation* on W associated with the modal operator $\Diamond_i$, for $1 \leq i \leq n$. A *(Kripke) model* $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ is a frame $\mathfrak{F} = (W, R_1, \dots, R_n)$ together with a propositional valuation $\mathfrak{V} : \textsc{prop} \to 2^W$ that assigns a set of possible worlds to each propositional variable $p_j \in \textsc{prop}$ in which p_j is *satisfied*. We say that $\mathfrak{F}$ is the *underlying frame* of $\mathfrak{M}$. We extend the notion of *satisfiability* to all modal formulas of $\mathcal{ML}_n$ with the following schema:

$$
\begin{aligned}
\mathfrak{M}, w \models p_j &\quad\iff\quad w \in \mathfrak{V}(p_j),\\
\mathfrak{M}, w \models \neg\varphi &\quad\iff\quad \mathfrak{M}, w \not\models \varphi,\\
\mathfrak{M}, w \models \varphi_1 \wedge \varphi_2 &\quad\iff\quad \mathfrak{M}, w \models \varphi_1 \text{ and } \mathfrak{M}, w \models \varphi_2,\\
\mathfrak{M}, w \models \Diamond_i\varphi &\quad\iff\quad \exists v \in W;\ wR_iv \text{ and } \mathfrak{M}, v \models \varphi,
\end{aligned}
$$

for $p_j \in \textsc{prop}$ and $1 \leq i \leq n$.

Given a modal formula $\varphi \in \mathcal{ML}_n$ and a frame $\mathfrak{F} = (W, R_1, \dots, R_n)$, we say that φ is:

- *satisfiable* in $\mathfrak{F}$ if $\mathfrak{M}, w \models \varphi$, for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ and some $w \in W$,
- *valid* in $\mathfrak{F}$ if $\mathfrak{M}, w \models \varphi$, for every model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ and every $w \in W$,

It follows that the above notions are dual, in the sense that $\neg\varphi$ is satisfiable in $\mathfrak{F}$ if and only if φ is not valid in $\mathfrak{F}$. We say that $\mathfrak{F}$ is a *frame for* φ, written $\mathfrak{F} \models \varphi$, whenever φ is valid in $\mathfrak{F}$.

Given a modal logic L we may define the class $\mathsf{Fr}\, L$ of all frames for L. That is to say,

$$\mathsf{Fr}\, L \ := \ \{\mathfrak{F} : \mathfrak{F} \models \varphi \text{ for all } \varphi \in L\}.$$

Associated with L, is the problem of deciding whether an arbitrary finite frame belongs to $\mathsf{Fr}\, L$; appropriately named the *finite frame problem for* L. The task is reduced to triviality whenever L is finitely axiomatisable, since we may simply check that each of the finitely many axioms are validated, but proves non-trivial whenever L is non-finitely axiomatisable [127]. Each of the logics described above in Table 2.3 is finitely axiomatisable and the frames which validates their theorems correspond to the following 'mathematically appealing' properties, described below in Table 2.4.

K	the class of all frames
T	the class of reflexive frames
Alt	the class of all partial functions
K4	the class of all transitive frames
S4	the class of all reflexive, transitive frames
S5	the class of equivalence relations
K4.3	the class of transitive, weakly-connected frames
S4.3	the class of reflexive, transitive, weakly-connected frames
wK4	the class of all weakly-transitive frames
wK5	the class of weakly-Euclidean frames
Diff	the class of symmetric, weakly-Euclidean frames
GL	the class of Noetherian strict partial orders
GL.3	the class of Noetherian strict linear orders
Grz	the class of Noetherian (reflexive) partial orders
Grz.3	the class of reflexive reflexive linear orders

Table 2.4: Some common Kripke complete logics and their respective frames.

Somewhat conversely, let $\mathcal{C}$ be some (non-empty) class of Kripke frames. Then we may define the logic *characterised* by $\mathcal{C}$, to be the set $\mathsf{Log}(\mathcal{C})$ of all modal formulas that are valid in every frame belonging to $\mathcal{C}$. That is to say,

$$\mathsf{Log}(\mathcal{C}) \;:=\; \{\varphi \in \mathcal{ML} \;:\; \mathfrak{F} \models \varphi \text{ for all } \mathfrak{F} \in \mathcal{C}\}.$$

It is straightforward to check that $\mathsf{Log}(\mathcal{C})$ contains all propositional tautologies and is closed under the each of the deduction rules given in Table 2.2.

We say that a given modal logic L is *sound with respect to* $\mathcal{C}$ if $\mathfrak{F} \models \varphi$ for all $\varphi \in L$ and all $\mathfrak{F} \in \mathcal{C}$; that is to say, that $L \subseteq \mathsf{Log}(\mathcal{C})$. Meanwhile, we say that L is *complete with respect to* $\mathcal{C}$ if $\varphi \in L$ whenever $\mathfrak{F} \models \varphi$ for all $\mathfrak{F} \in \mathcal{C}$; that is to say, that $\mathsf{Log}(\mathcal{C}) \subseteq L$. We say that a logic L is *Kripke complete*[†] if it is both sound and complete with respect to some class of Kripke frames $\mathcal{C}$; that is to say, whenever $L = \mathsf{Log}(\mathcal{C})$.

Note that the class of frames with respect to which a logic is sound and complete, need not be unique. Consequently, while it is always true that $\mathsf{Log}(\mathsf{Fr}\, L) = L$, it may arise that an intended class of frames $\mathcal{C}$ is *strictly* subsumed by $\mathsf{Fr}\,\mathsf{Log}(\mathcal{C})$.

An n-modal logic L is said to be *canonical* if all of its theorems are valid in its *canonical frame* $\mathfrak{F}^L = (W^L, R_1^L, \ldots, R_n^L)$, where,

- W^L is the set of all maximally consistent subsets of $\mathcal{ML}_n$ containing L,
- For all $i = 1, \ldots, n$, we have that $uR_i^L v$ if and only if $\Diamond_i \psi \in u$ for all $\psi \in v$.

The properties of the canonical frame are such that every non-theorem of L is refutable in $\mathfrak{F}^L$ under the *canonical valuation* $\mathfrak{V}^L$, whereby $u \in \mathfrak{V}^L(p)$ if and only if $p \in u$. It follows that every canonical modal logic is Kripke complete.

Proposition 2.1. *Every canonical logic is Kripke complete.*

Dual to the decision problem for L, described above, is the *satisfiability problem*, which asks whether there is an effective procedure for distinguishing those formulas satisfiable in some frame for L, from those not satisfiable in any frame for L. From the foregoing we see that, for Kripke complete modal logics, the decision problem belongs to the complexity class $\mathscr{C}$ if and only if the satisfiability problem belongs to the class $\textsc{co}\mathscr{C}$, to use the standard terminology outlined in Appendix A.

A typical approach to demonstrating decidability for modal logics — and one that will be used frequently, herein — is to establish some upper bound on the size of the search space in which one seeks to find a refuting model for every non-theorem. It follows from the Löwenheim-Skolem Theorem that if L is characterised by some first-order definable class of frames then L is sound and complete with respect to its *countable* frames [14]. However, what we are most interested in are those logics that are sound and complete with respect to their *finite* frames.

[†]The first examples of modal logics that are not Kripke complete were given by Thomason [124] and Fine [34]. Perhaps the simplest example of such a Kripke incomplete logic is the logic $\mathbf{K} + \Box\Diamond\top \to \Box\big(\Box(\Box p \to p) \to p\big)$, given by van Benthem [128].

Definition 2.2. Suppose that L is a modal logic. We say that L has the *finite model property (fmp)* if every $\varphi \notin L$ can be refuted in some *finite* frame for L. More specifically, given a recursive function $f : \omega \to \omega$, we say that L has the *f-size fmp* if every $\varphi \notin L$ can be refuted in some frame for L, whose size does not exceed $f(n)$ where $n = |\mathrm{sub}(\varphi)|$ is the size of φ. Should such a recursive function exist, we say that L has the *effective fmp*. In particular, L has the *polysize fmp* if f can be chosen to be a polynomial, and the *exponential fmp* if f can be chosen to be a singly exponential function.

If the finite frame problem for L is decidable, then any recursive bound on the size of the models for L, in which any non-theorem can be refuted, provides us with an effective decision procedure for deciding theoremhood; given an arbitrary modal formula φ, it is enough to enumerate all such 'small' models, based on frames for L, and sequentially search for a refuting model.

Even without a recursive bound on the size of the refuting models, the finite model property provides a co-recursively enumerable upper bound on the complexity of the decision problem for L, whenever the class of finite frames for L is recursive. Indeed, it follows that if L is finitely axiomatisable and has the finite model property, then L is decidable.

It should be noted that neither the finite model property nor a finite axiomatisation, alone, is sufficient to guarantee decidability [127, 65], nor are they both necessary [37, 64, 25].

2.2 Truth-preserving Transformations

Two frames are said to be *(modally) equivalent* if they validate the same modal formulas. We conclude this chapter with a description of some well-known model transformations that preserve modal equivalence, and which will be used frequently hereafter.

Generated Subframes

A frame $\mathfrak{F} = (W, R_1, \ldots, R_n)$ is said to be a *generated subframe* of $\mathfrak{F}' = (W', R'_1, \ldots, R'_n)$ if: (i) $W \subseteq W'$, (ii) if $w \in W$ and $wR'_i v$, for some $1 \leq i \leq n$, then $v \in W$, and (iii) $R_i = R'_i \cap (W \times W)$, for all $1 \leq i \leq n$.

We say that $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ is a *generated submodel* of $\mathfrak{M}' = (\mathfrak{F}', \mathfrak{V}')$ if $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}'$ and $\mathfrak{V}(p) = \mathfrak{V}'(p) \cap W$, for all propositional variables $p \in \textsc{prop}$.

Proposition 2.3. *Let $\mathfrak{M}'$ be a generated submodel of $\mathfrak{M}$. Then $\mathfrak{M}, w' \models \varphi$ if and only if $\mathfrak{M}', w' \models \varphi$, for all $w' \in W'$ and $\varphi \in \mathcal{ML}_n$.*

Proof. See Propositions 2.6 of [14]. □

A frame $\mathfrak{F}$ is said to be *rooted* or *point-generated* if there is some $r \in W$, called a *root of* $\mathfrak{F}$, such that $\mathfrak{F}$ is the smallest generated subframe containing r. It follows from Proposition 2.3 that every Kripke complete modal logic is complete with respect to its *rooted frames.*

P-morphisms

A *pseudo-epimorphism*, or more succinctly a *p-morphism*† between two frames, $\mathfrak{F} = (W, R_1, \ldots, R_n)$ and $\mathfrak{F}' = (W', R'_1, \ldots, R'_n)$, is a function $f : W \to W'$ such that, for all $w, v \in W$:

- if $wR_i v$ then $f(w)R'_i f(v)$,
- if $f(w)R'_i v$ then there is some $u \in W$ such that $wR_i u$ and $f(u) = v$.

for $1 \leq i \leq n$. If f is surjective then $\mathfrak{F}'$ is said to be a *p-morphic* image of $\mathfrak{F}$.

A p-morphism between models $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ and $\mathfrak{M}' = (\mathfrak{F}', \mathfrak{V}')$ is a p-morphism between $\mathfrak{F}$ and $\mathfrak{F}'$ such that if $w \in \mathfrak{V}(p)$ then $f(w) \in \mathfrak{V}'(p)$, for all propositional variables $p \in \textsc{prop}$. We say that $\mathfrak{M}'$ is a p-morphic image of $\mathfrak{M}$ whenever f is surjective.

†Sometimes referred to as *bounded morphisms* in the literature.

Proposition 2.4. *Let $\mathfrak{M}'$ be the p-morphic image of $\mathfrak{M}$ given by the p-morphism f. Then $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M}', f(w) \models \varphi$, for all $w \in W$ and $\varphi \in \mathcal{ML}_n$.*

Proof. See Propositions 2.14 of [14]. $\square$

It follows from Proposition 2.4 that if L is a Kripke complete modal logic and $\mathcal{C}$ comprises some class of frames such that every frame for L is the p-morphic image of some frame belonging to $\mathcal{C}$, then L is complete with respect to $\mathcal{C}$. Hence, together with the foregoing observations, any non-theorem of L must be refuted in some countable, rooted frame belonging to $\mathcal{C}$ — an observation that will be exploited frequently, hereafter.

While the above transformations will appear frequently throughout this thesis, the following transformations will be used sparingly, and their details are included here only for the sake of completeness.

Ultraproducts

Given a non-empty index set I, we define an *ultrafilter over* I to be any set $U \subseteq 2^I$ such that: (i) $\emptyset \notin U$, (ii) if $X \in U$ and $Y \in U$ then $X \cap Y \in U$, (iii) if $X \in U$ and $X \subseteq Y$ then $Y \in U$, (iv) if $X \notin U$ then $(I - X) \in U$.

Given a collection of frames $\mathfrak{F}_i = (W_i, R_i^1, \ldots, R_i^n)$, for $i \in I$, and an ultrafilter U over I, let $C = \prod_{i \in I} W_i$, be the Cartesian product, comprising all those functions $f : I \to \bigcup_{i \in I} W_i$, such that $f(i) \in W_i$, for all $i \in I$. We define the equivalence relation $\sim_U$ on C, by taking

$$f \sim_U g \qquad \Longleftrightarrow \qquad \{i \in I : f(i) = g(i)\} \in U$$

for all $f, g \in C$. We then define the *ultraproduct* of $\{\mathfrak{F}_i : i \in I\}$ to be the frame $\prod_U \mathfrak{F}_i = (W_U, R_U^1, \ldots, R_U^n)$, where

- $W_U = \{[f] : f \in C\}$, where $[f]$ is the $\sim_U$-equivalence class containing f,
- $[f] R_U^k [g]$ if and only if $\{i \in I : f(i) R_i^k g(i)\} \in U$, for $1 \leq k \leq n$.

We say that a class of frames $\mathcal{C}$ is closed under ultraproducts if $\prod_U \mathfrak{F}_i \in \mathcal{C}$ whenever $\mathfrak{F}_i \in \mathcal{C}$, for all $i \in I$ and every ultrafilter U over I.

Ultrafilter Extensions

Given a frame $\mathfrak{F} = (W, R_1, \ldots, R_n)$, we may define the *ultrafilter extension of* $\mathfrak{F}$ to be the frame $\mathfrak{Ue}\mathfrak{F} = (Uf(W), R_1^{ue}, \ldots, R_n^{ue})$, where

- $Uf(W)$ is the set of all ultrafilters over W,
- $uR_i^{ue}v$ if and only if $m_i(X) \in u$, for all $X \in v$, where

$$m_i(X) = \{w \in W : wR_iw' \text{ for some } w' \in X\}$$

  collects all those worlds from which X is R_i-accessible, for $i = 1, \ldots, n$.

It is well-known that the *principal ultrafilters*, of the form $\pi_w = \{X \subseteq W : w \in X\}$, identify an isomorphic copy of $\mathfrak{F}$ as a submodel of $\mathfrak{Ue}\mathfrak{F}$ via the embedding $\iota : w \mapsto \pi_w$. Moreover, the *only* ultrafilters of a finite set are those principal ultrafilters, and therefore we find that $\mathfrak{F}$ is isomorphic to $\mathfrak{Ue}\mathfrak{F}$, whenever $\mathfrak{F}$ is finite.

It follows from Proposition 2.4 that if $\mathfrak{F}$ is a p-morphic image of some $\mathfrak{G} \in \mathcal{C}$ then $\mathfrak{F}$ is a frame for $\mathsf{Log}(\mathcal{C})$. However, it is not that case that every frame for $\mathsf{Log}(\mathcal{C})$ is the p-morphic image of some frame $\mathfrak{G} \in \mathcal{C}$. The more general relationship between logics and frames is established by the following result of Kurucz [75].

Theorem 2.5 (Kurucz [75])**.** *Let $\mathcal{C}$ be any class of frames closed under ultraproducts and point-generated subframes. Then, for every rooted frame $\mathfrak{F}$,*

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \Longleftrightarrow \qquad \mathfrak{Ue}\mathfrak{F} \textit{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}.$$

Proof. See Corollary 2.5 of [75]. $\square$

Given that $\mathfrak{Ue}\mathfrak{F}$ is isomorphic to $\mathfrak{F}$, whenever $\mathfrak{F}$ is finite, we have that every *finite* frame is a frame for $\mathsf{Log}(\mathcal{C})$ if and only if it is the p-morphic image of a frame belonging to $\mathcal{C}$.

Chapter 3

Combining Modal Logics

3.1 Fusions of Modal Logics

Suppose that L_1 is an n-modal logic and that L_2 is an m-modal logic. We define a translation $s_n : \mathcal{ML}_m \to \mathcal{ML}_{(n+m)}$, which replaces every occurrence of $\Diamond_i$ with $\Diamond_{i+n}$, for all $i < m$. Take $L_2' = \{s_n(\varphi) : \varphi \in L_2\}$ to be the set of $(n+m)$-modal formulas obtained by applying this translation to all theorems of L_2. Then L_2' contains none of the modal operators $\Diamond_1, \ldots, \Diamond_n$.

We define the *fusion* or *independent join* of L_1 and L_2, written $L_1 \otimes L_2$, to be the smallest modal logic subsuming $L_1 \cup L_2'$, where L_1 and L_2' contain no modal operators in common. We note that if Γ_i axiomatises L_i, for $i = 1, 2$, then $\Gamma_1 \cup \Gamma_2'$ axiomatises $L_1 \otimes L_2$, where $\Gamma_2' = \{s_n(\varphi) : \varphi \in \Gamma_2\}$. Consequently, $L_1 \otimes L_2$ is finitely axiomatisable whenever both L_1 and L_2 are finitely axiomatisable.

Suppose that $\mathcal{C}_1$ is some class of n-modal frames and $\mathcal{C}_2$ is some class of m-modal frames, both closed under disjoint unions and isomorphic copies. Then the fusion $\mathcal{C}_1 \otimes \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ is taken to be the class of all $(n+m)$-frames of the form

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = (W, R_1, \ldots, R_n, S_1, \ldots, S_m),$$

where $\mathfrak{F}_1 = (W, R_1, \ldots, R_n) \in \mathcal{C}_1$ and $\mathfrak{F}_2 = (W, S_1, \ldots, S_m) \in \mathcal{C}_2$.

The following theorem of Kracht and Wolter, shows that the process of taking fusions commutes with the formation of logics.

Theorem 3.1 (Kracht-Wolter [67]). *Suppose that $\mathcal{C}_h$ and $\mathcal{C}_v$ are two classes of Kripke frames. Then* $\mathsf{Log}(\mathcal{C}_h) \otimes \mathsf{Log}(\mathcal{C}_v) = \mathsf{Log}(\mathcal{C}_h \otimes \mathcal{C}_v)$. *In particular, if L_h and L_v are two Kripke complete modal logics, then*

$$L_1 \otimes L_2 = \mathsf{Log}\big(\mathsf{Fr}\, L_1 \otimes \mathsf{Fr}\, L_2\big).$$

Fusions are the most general method for combing modal logics; with their modalities interacting in only the most trivial of cases. As such, the complexity of their decision problems do not far exceed that of their constituent parts. In particular, $L_1 \otimes L_2$ is decidable whenever both L_1 and L_2 are decidable [133, 121].

3.2 Products of Modal Logics

The product construction was first described by Segerberg [113] who introduced the semantics for the case of two interacting **S5**-modalities. This construction was generalised by Shehtman [115] to include products of arbitrary modal logics, introducing the notation used here. Since their inception, many-dimensional modal logics have been studied extensively [38, 73, 39, 40, 85]. However, while much is settled in dimension three and higher, there remains much to be explored with regard to two-dimensional modal logics.

Definition 3.2. Given two frames $\mathfrak{F}_h = (W_h, R_h^1, \ldots, R_h^n)$ and $\mathfrak{F}_v = (W_v, R_v^1, \ldots, R_v^m)$, we define their (two-dimensional) *product frame* to be the $(n+m)$-frame

$$\mathfrak{F}_h \times \mathfrak{F}_v \;:=\; \big(W_h \times W_v, \bar{R}_h^1, \ldots, \bar{R}_h^n, \bar{R}_v^1, \ldots, \bar{R}_v^m\big),$$

where $W_h \times W_v = \{(x, y) : x \in W_h \text{ and } y \in W_v\}$ is the Cartesian product of W_h and W_v and, for all $x, x' \in W_h$ and $y, y' \in W_v$, we have that

$$(x,y)\bar{R}_h^i(x',y') \iff xR_h^i x' \text{ and } y = y' \quad \text{for } 1 \le i \le n,$$
$$(x,y)\bar{R}_v^j(x',y') \iff x = x' \text{ and } yR_v^j y' \quad \text{for } 1 \le j \le m.$$

The subscripts h and v betray the geometric intuition behind this construction, illustrated in Figure 3.1; with the $\bar{R}_h^i$ denoting the *'horizontal'* accessibility relations and the $\bar{R}_v^j$ denoting the *'vertical'* accessibility relations.

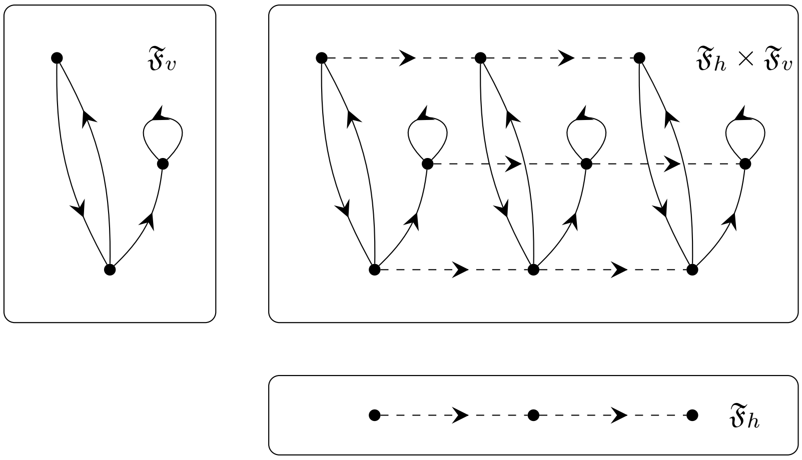


Figure 3.1: Illustration of the product construction.

It is well-established that this construction commutes with taking generated subframes and p-morphic images, in the sense detailed by the following proposition of Shehtman.

Proposition 3.3 (Shehtman [115]). *Let $\mathfrak{F}_i, \mathfrak{G}_i$ be frames, for $i = h, v$ then the following hold:*

(i) *If $\mathfrak{F}_i$ is a generated subframe of $\mathfrak{G}_i$, for $i = h, v$, then $\mathfrak{F}_h \times \mathfrak{F}_v$ is a generated subframe of $\mathfrak{G}_h \times \mathfrak{G}_v$,*

(ii) *If $\mathfrak{F}_i$ is a p-morphic image of $\mathfrak{G}_i$, for $i = h, v$, then $\mathfrak{F}_h \times \mathfrak{F}_v$ if a p-morphic image of $\mathfrak{G}_h \times \mathfrak{G}_v$.*

Proof. See Lemma 1 of [115]. $\square$

Given two classes of frames $\mathcal{C}_h$ and $\mathcal{C}_v$, we may define their *product class* $\mathcal{C}_h \times \mathcal{C}_v$ to be the class of all product frames $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_i \in \mathcal{C}_i$, for $i = h, v$. For modal logics L_h and L_v, we define their *product logic* $L_h \times L_v$ by taking

$$L_h \times L_v \ := \ \mathsf{Log}\big(\mathsf{Fr}\, L_h \times \mathsf{Fr}\, L_v\big).$$

That is to say, the logic characterised by all those product frames $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_i \in \mathsf{Fr}\, L_i$ is a frame for L_i, for $i = h, v$.

It is straightforward to check that the product construction also commutes with the formation of ultraproducts [75]. It follows that if both $\mathcal{C}_h$ and $\mathcal{C}_v$ are closed under point-generated subframes and ultraproducts, then so is the product class $\mathcal{C}_h \times \mathcal{C}_v$. Hence we have the following corollary of Theorem 2.5.

Theorem 3.4 (Kurucz [75]). *For $i = h, v$, let $\mathcal{C}_i$ be a class of frames closed under ultraproducts and point-generated subframes. Then, for every rooted frame $\mathfrak{F}$,*

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v) \iff \mathfrak{Ue}\mathfrak{F} \text{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}_h \times \mathcal{C}_v.$$

Proof. See Theorem 2.10 of [75] □

3.2.1 Complexity of Products

Unlike fusions, whose decision problems are decidable whenever their constituent parts are decidable, the complexity of products may often vastly exceed that of their constituent parts. Some notable examples include $\mathbf{K4}\times\mathbf{K4}$ and $\mathbf{K4.3}\times\mathbf{K4.3}$, whose decision problems are both undecidable [41, 102], despite the relatively modest complexity of both $\mathbf{K4}$ and $\mathbf{K4.3}$ — PSPACE-complete and NP-complete, respectively [78, 91].

However, for product of logics, whose frames describe a first-order definable class, the explosion of complexity is curtailed by the following general theorem of Gabbay and Shehtman.

Theorem 3.5 (Gabbay-Shehtman [39]). *Let $\mathcal{C}_h$ and $\mathcal{C}_v$ be any classes of frames definable by a recursive set of first-order formulas in the (purely relational) language having equality and a binary predicate symbol for each modal operator. Then the decision problem for $\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)$ is recursively enumerable.*

Proof. See Theorem 5.5 of [39]. □

In particular, the decision problem for $L_h \times L_v$ is recursively enumerable, whenever $\mathsf{Fr}\, L_i$ describes a first-order definable class of frames, for $i = h, v$. However, while this places a recursively enumerable upper bound on the complexity of the decision problem, it leaves open the matter of decidability.

A typical approach to demonstrating decidability for product logics, as with regular one-dimensional logics, is to establish some upper bound on the size of search space in which one seeks to find a refuting model to every non-theorem. However, while product logics are, by definition, characterised by their product frames, this does not preclude the possibility of there being some *abstract* frames, validating the same formulas, yet not being isomorphic to any product frame. Consider, for example, the frame $(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2)$ which despite being a frame for $\mathbf{S5} \times \mathbf{S5}$ — being, as it is, a p-morphic image of the product frame $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \mathbb{Z}^2)^\dagger$ — is not isomorphic to any product frame for $\mathbf{S5} \times \mathbf{S5}$.

It may not always be apparent what such abstract frames actually look like, or whether it is, indeed, possible to enumerable them, as we require. For this reason it is tempting to consider the following specialisation of, what may be termed, the *abstract finite model property*, defined above in Definition 2.2

Definition 3.6. A product logic $L_h \times L_v$ is said to have the *finite product model property*, or *product fmp* for brevity, if every $\varphi \notin L_h \times L_v$ can be refuted in some finite *product* frame for $L_h \times L_v$. Given a recursive function $f : \omega \to \omega$, we say that $L_h \times L_v$ has the *f-size product fmp* if every $\varphi \in L_h \times L_v$ can be refuted in some product frame for $L_h \times L_v$, whose size does not exceed $f(n)$, where $n = |\mathrm{sub}(\varphi)|$ is the size of φ. In particular, $L_h \times L_v$ has the *polysize product fmp* if f can be chosen to be a polynomial, and the *exponential product fmp* if f can be chosen to be a singly exponential function.

Clearly the definition of the product fmp demands more specificity than that of the abstract fmp. Thus, any logic possessing the former must, necessarily, possess the latter. Interestingly, the converse does not hold, as may be evidenced by $\mathbf{K} \times \mathbf{K4}$, which despite having the abstract fmp, does not possess the *product* fmp [116].

To add a further level of granularity, we introduce the notion of the *square product fmp* as a further specialisation of the product fmp.

Definition 3.7. A product frame $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_i = (W_i, R_i)$ for $i = h, v$, is said to be a *square product frame*‡, whenever $|W_h| = |W_v|$.

We say that a bimodal logic L has the *finite square product model property*, or *square product fmp*, if every $\varphi \notin L$ can be refuted in some finite *square* product frame for L.

†Take, for example, the p-morphism $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(n, m) = n + m$.

‡Note that this definition differs from that considered in [39], in that we do not insist on the relations R_h and R_v being identical.

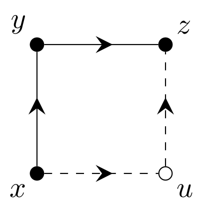
Clearly this definition demands more specificity than that of both the product fmp and abstract fmp, and thus, any logic possessing the square product fmp must, necessarily, possess both the product fmp and abstract fmp.

As we shall see in Section 6.2, the converse does not hold here either, for there exist product logics characterised by their finite product frames that are not characterised by their finite *square* product frames.

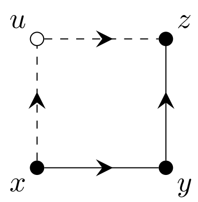
3.2.2 Axiomatising Products

It is straightforward to see that each product logic $L_h \times L_v$ is a normal extension of the fusion $L_h \otimes L_v$. However, since all product logics are characterised by some class of product frames, they admit additional theorems not common to their respective fusions. The three most notable properties that are valid in all product frames are the following:

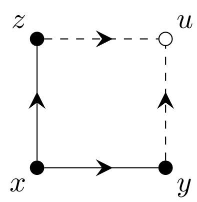
- *Left-commutativity:* $\forall x \forall y \forall z \ \big(xR_v y \wedge yR_h z \to \exists u(xR_h u \wedge uR_v z)\big)$,
- *Right-commutativity:* $\forall x \forall y \forall z \ \big(xR_h y \wedge yR_v z \to \exists u(xR_v u \wedge uR_h z)\big)$,
- *Church-Rosser property:* $\forall x \forall y \forall z \ \big(xR_h y \wedge xR_v z \to \exists u(yR_v u \wedge zR_h u)\big)$.



Left-commutativity



Right-commutativity



Church-Rosser

Each of these properties is *modally definable*, with the following formulas valid in precisely those frames that are left-commutative and right-commutative, respectively,

$$(com^l) \ := \ \Diamond_v \Diamond_h p \to \Diamond_h \Diamond_v p \qquad \text{and} \qquad (com^r) \ := \ \Diamond_h \Diamond_v p \to \Diamond_v \Diamond_h p,$$

while the following formula is valid in precisely those frames that satisfy the Church-Rosser property,

$$(chr) \ := \ \Diamond_v \Box_h p \to \Box_h \Diamond_v p.$$

We define the *commutator* $[L_h, L_v]$ of L_h and L_v to be the smallest modal logic containing $L_h \otimes L_v$ together with the axioms (com^l), (com^r), and (chr). It follows from a standard classification result of Sahlqvist [104] that each of these axioms is canonical. Hence, by Proposition 2.1, we have the following completeness result for commutators of canonical logics.

Theorem 3.8. *If both L_h and L_v are canonical then their commutator $[L_h, L_v]$ is also canonical, and thus Kripke complete.*

It is then straightforward to check that the commutator $[L_h, L_v]$ of two canonical logics is complete with respect to the class of all bimodal frames of the form (W, R_h, R_v), where $(W, R_i) \in \mathsf{Fr}\, L_i$ for $i = h, v$.

In [39], it is shown that, for a large class of product logics, these axioms are sufficient to describe the full product. Two logics L_h and L_v are said to be *product matching* whenever $L_h \times L_v = [L_h, L_v]$.

Definition 3.9. A unimodal logic L is said to be *Horn-axiomatisable* if it is axiomatisable by a set of formulas Γ, such that each $\varphi \in \Gamma$ is sound and complete with respect to some class of frames satisfying a first-order property of the form

$$\forall x \forall y \forall \vec{z}\ \big(\psi(x, y, \vec{z}) \to xRy\big)$$

where $\psi(x, y, \vec{z})$ is built up from atoms using only the connectives $\wedge$ and $\vee$. Logics that cannot be axiomatised by *any* such set of *Horn-formulas* are said to be *non-Horn-axiomatisable.*

Theorem 3.10 (Gabbay-Shehtman [39])**.** *Let L_h and L_v be any two Kripke complete, Horn-axiomatisable modal logics. Then L_h and L_v are product matching.*

In particular, every frame for $[L_h, L_v]$ is the p-morphic image of a product frame $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_h \in \mathsf{Fr}\, L_h$ and $\mathfrak{F}_v \in \mathsf{Fr}\, L_v$.

Many of the most familiar modal logics, such as $\mathbf{K}, \mathbf{T}, \mathbf{D}, \mathbf{K4}, \mathbf{S4}$ and $\mathbf{S5}$ are Horn-axiomatisable, and so their products fall within the remit of Theorem 3.10.

Chapter 4

Products and Two-Variable First-order Logics

The connections between modal logics and various fragments of first-order logic have been extensively studied [132, 99, 71, 130]. Perhaps the simplest such connection is given by the standard translation, which interprets the unimodal logic **K** as a decidable fragment of first-order logic — the so-called *guarded fragment*. Variations on this standard translation yield more specialised embeddings of modal logics within other fragments of first-order logic.

In Section 4.1, we give a brief overview of some standard definitions and results pertaining to first-order logic. In particular, we introduce the two-variable fragment $\mathcal{L}^2$ and the two-variable fragment with counting quantifiers $\mathcal{C}^2$, both known to be decidable. A full treatment of these topics can be found in [31, 17], for example.

Section 4.2 provides an overview of von Wright's 'logic of elsewhere' **Diff**, and describes the relationships it shares with **S5**.

Finally, in Section 4.3, we describe the standard translation and variations thereof, which embed the product logic **S5** $\times$ **S5** within the two-variable fragment of first-order logic, and the product **Diff** $\times$ **Diff** within the two-variable fragment equipped with the counting quantifiers $\exists_{>m}x$ and $\exists_{>m}y$, for $m = 0, 1$.

4.1 First-order Logic

Let $\mathcal{L}_{\approx}$ denote the first-order language (with equality) comprising a countably infinite set of predicate symbols $\textsc{pred} = \{P_0, P_1, \dots\}$, a countable set of first-order variables $Var = \{x_0, x_1, \dots\}$, logical symbols for *negation* $\neg$ and *conjunction* $\wedge$, an equality symbol $\approx$, a single *existential quantifier* $\exists$, and punctuation. With each predicate symbol $P_j \in \textsc{pred}$, we associate an *arity* n, calling P_j *monadic* whenever $n = 1$, and *binary* whenever $n = 2$. The *formulas* of $\mathcal{L}_\approx$ are defined according to the grammar:

$$\varphi \quad ::= \quad x_1 \approx x_2 \;\mid\; P_j(x_1, \ldots, x_n) \;\mid\; \neg\varphi \;\mid\; (\varphi_1 \wedge \varphi_2) \;\mid\; \exists x\ \varphi$$

where $P_j \in \textsc{pred}$ is an n-ary predicate symbol. The *length* of each formula is defined to be the number of symbols it comprises, with variables given a binary encoding. We make use of the standard abbreviations for *disjunction* $\vee$, *implication* $\rightarrow$, *equivalence* $\leftrightarrow$, and the *universal quantifier* $\forall x\ \varphi := \neg\exists x\ \neg\varphi$.

Formulas of $\mathcal{L}_\approx$ are interpreted over *first-order structures* $\mathfrak{A} = (A, (\cdot)^{\mathfrak{A}})$, where A is a *domain* of interpretation and $(\cdot)^{\mathfrak{A}}$ is an *interpretation function* mapping each n-ary predicate symbol $P_j \in \textsc{pred}$ to some subset $P_j^{\mathfrak{A}} \subseteq A^n$. A *variable assignment* h is a function mapping Var into A [31].

Satisfiability of first-order formulas is defined inductively as follows:

$$\begin{array}{lcl}
\mathfrak{A} \models^h x_1 \approx x_2 & \iff & h(x_1) = h(x_2) \\
\mathfrak{A} \models^h P_j(x_1, \ldots, x_n) & \iff & \big(h(x_1), \ldots, h(x_n)\big) \in P_j^{\mathfrak{A}} \\
\mathfrak{A} \models^h \neg\varphi & \iff & \mathfrak{A} \not\models^h \varphi \\
\mathfrak{A} \models^h \varphi_1 \wedge \varphi_2 & \iff & \mathfrak{A} \models^h \varphi_1 \text{ and } \mathfrak{A} \models^h \varphi_2
\end{array}$$

and

$$\mathfrak{A} \models^h \exists x\ \varphi \qquad \iff \qquad \big|\{a \in A \;:\; \mathfrak{A} \models^{h(a/x)} \varphi\}\big| > 0$$

where $h(a/x) : Var \rightarrow A$ is the variable assignment that agrees with h on all variables except x, for which it assigns the value $a \in A$, and $|X|$ denotes the cardinality of the set X.

We say that a first-order formula φ is a *theorem* of $\mathcal{L}_\approx$ if $\mathfrak{A} \models^h \varphi$, for all first-order structures $\mathfrak{A}$ and all variable assignments h. With each fragment of $\mathcal{L}_\approx$, we associate its *validity problem*, which asks whether an arbitrary first-order formula of the given fragment is a theorem of $\mathcal{L}_\approx$.

It was proved by Church [22] and Turing [126] that the validity problem for first-order logic, in its full generality, is undecidable. This remains the case, even if we restrict attention to the three-variable fragment [122, 123]. However there exists many natural fragments for which the validity problem is decidable (see [17] for details and references).

Among which, are the two-variable fragment $\mathcal{L}^2_\approx$ and the equality-free two-variable fragment $\mathcal{L}^2$. It was proved by Mortimer that $\mathcal{L}^2_\approx$ enjoys the finite model property and is, therefore, decidable [88]†. The double exponential bound given by Mortimer was later improved upon by Grädel et al. [51] who provided the optimal coNExpTime upper bound.

Theorem 4.1 (Grädel et al. [51])**.** *The validity problem for $\mathcal{L}^2_\approx$ is* coNExpTime*-complete.*

Counting Quantifiers

The language $\mathcal{C}_\approx$ extends $\mathcal{L}_\approx$ with the addition of infinitely many counting quantifiers of the form $\exists_{>m}$, for each $m < \omega$. The *formulas* of $\mathcal{C}_\approx$ are defined according to the grammar:

$$\varphi \quad ::= \quad x_1 \approx x_2 \ \mid \ P_j(x_1, \ldots, x_n) \ \mid \ \neg\varphi \ \mid \ (\varphi_1 \wedge \varphi_2) \ \mid \ \exists_{>m} x \ \varphi$$

where P is an n-ary predicate symbol, $x, x_1, \ldots, x_n \in Var$ are first-order variables, and $m < \omega$. The *size* of a formula φ is taken to mean the number of symbols it comprises, where m is given a binary encoding. We make use of the same abbreviations as above as well as introducing the following abbreviations new to $\mathcal{C}_\approx$:

$$\exists x \ \varphi \ := \ \exists_{>0} x \ \varphi, \qquad \forall x \varphi \ := \ \neg\exists x \neg\varphi, \qquad \exists_{<m} x \ \varphi \ := \ \neg\exists_{>m-1} x \ \varphi.$$

Satisfiability of $\mathcal{C}_\approx$ formulas is defined over first-order structures as it is for formulas of $\mathcal{L}_\approx$, with the addition that

$$\mathfrak{A} \models^h \exists_{>m} x \ \varphi \qquad \Longleftrightarrow \qquad \big|\{a \in A \ : \ \mathfrak{A} \models^{h(a/x)} \varphi\}\big| > m,$$

for all $m < \omega$. We say that a first-order formula φ is a *theorem* of $\mathcal{C}_\approx$ if $\mathfrak{A} \models^h \varphi$, for all first-order structures $\mathfrak{A}$ and all variable assignments h. With each fragment of $\mathcal{C}_\approx$, we associate its *validity problem*, which asks whether an arbitrary first-order formula of the

†This result is sometimes attributed to Scott [110], who provided a normal form for $\mathcal{L}^2_\approx$ that is subsumed by the so-called *Gödel fragment with equality*, containing those formulas whose prefix normal form has the quantifier pattern $\forall^2\exists^*$. However, Gödel only proved this fragment to be decidable in the absence of equality, wrongly asserting this result could be extended to the full class with equality; a claim only later noted to be false by Goldfarb [48] in 1984. See [17, page 404].

given fragment is a theorem of $\mathcal{C}_{\approx}$.

In what follows, we will be particularly interested in the two-variable fragment $\mathcal{L}^2_{\approx}$ and the two-variable equality-free fragment $\mathcal{L}^2$. For while counting quantifiers can be expressed in $\mathcal{L}_{\approx}$, to do so requires arbitrarily many first-order variable that we do not have at our disposal in any finite variable fragment. Hence the two-variable fragment with counting quantifiers $\mathcal{C}^2$ is strictly more expressive than that of its counting-free counterpart $\mathcal{L}^2$.

Indeed, $\mathcal{C}^2_{\approx}$ does not share the same finite model property enjoyed by $\mathcal{L}^2$, as evidenced by the following formula [52]:

$$\theta_\infty \ := \ \forall x \exists y \ P(x,y) \wedge \forall y \exists_{\leq 1} x \ P(x,y) \wedge \exists y \forall x \ \neg P(x,y). \tag{4.1}$$

Despite being satisfiable, the above formula cannot be satisfied in any *finite* first-order structure. For suppose that θ_∞ is satisfiable in some first-order structure $\mathfrak{A} = (A, (\cdot)^{\mathfrak{A}})$. Then we may define a function $f : A \to A$ by choosing $f(a) \in \{b \in A : P^{\mathfrak{A}}(a,b)\}$ for all $a \in A$. Whatever our choice of f, we are assured that f is an injective, non-surjective mapping of A into A, and hence cannot be realised over any finite domain.

That being so, in [52], the authors show that, despite the lack of any finite model property, the validity problem for $\mathcal{C}^2_{\approx}$ (with equality) is, nonetheless, decidable. A non-deterministic double-exponential upper bound was provided in the same year by Pacholski et al. [94]. In the same paper, the authors also show that if we limit ourselves to the fragment containing only quantifiers from among $\{\exists_{>m} x, \exists_{>m} y : m = 0, 1\}$, then the validity problem can be decided in coNExpTime. The tight upper bound for the full two-variable fragment was later provided by Pratt-Hartmann [96, 97], who showed that the validity problem can still be decided in coNExpTime, even if we drop the restriction on the quantifiers involved.

Theorem 4.2 (Pratt-Hartmann [96]). *The validity problem for $\mathcal{C}^2_{\approx}$ is* coNExpTime-*complete.*

4.2 The Logic of 'Elsewhere'

In [131], von Wright introduced the logic of 'elsewhere', characterised by those frames whose accessibility relation connects all, and only those, distinct worlds. We say that $\mathfrak{F} = (W, R)$ is a *difference frame* if $R = \{(x, y) : x \neq y\}$, in which case we may abbreviate $\mathfrak{F}$ as $(W, \neq)$. Von Wright's logic is then defined, by taking,

$$\mathbf{Diff} := \mathsf{Log}\{\mathfrak{F} : \mathfrak{F} \text{ is a difference frame}\}.$$

Such frames closely resemble *universal frames*, whose accessibility relation connects *all* worlds. The class of all universal frames characterises Lewis's **S5** logic [79, 68]. Like **S5**, whose theorems are validated in frames other than those universal frames, there are frames for **Diff** that need not belong to the intended class of all difference frames. However, it was proved by Segerberg that every frame for **Diff** is both symmetric and *weakly-transitive*† [114]. A frame is said to be weakly-transitive if it satisfies the following first-order condition:

– *weak-transitivity:* $\forall x \forall y \forall z \, \big(xRy \wedge yRz \rightarrow (x = z \vee xRz)\big)$.

From this characterisation of the frames for **Diff**, Segerberg notes that **Diff** is the smallest unimodal logic containing the following axioms:

$$(B) := p \rightarrow \Box\Diamond p \qquad \text{and} \qquad (w4) := \Diamond\Diamond p \rightarrow (p \vee \Diamond p),$$

corresponding to symmetry and weak-transitivity, respectively.

From this initial characterisation by Segerberg, it is a straightforward exercise to show that every frame for **Diff** is both symmetric and *weakly-Euclidean*, where a frame is said to be weakly-Euclidean is it satisfies the following condition:

– *weak-Euclideanness:* $\forall x \forall y \forall z \, \big(xRy \wedge xRz \rightarrow (y = z \vee yRz)\big)$.

This characterisation provides a alternative axiomatisation for **Diff**, as the smallest unimodal logic containing the axioms:

$$(B) := p \rightarrow \Box\Diamond p \qquad \text{and} \qquad (w5) := \Diamond p \rightarrow \Box(p \vee \Diamond p),$$

corresponding to symmetry and weak-Euclideanness, respectively. We denote by **wK5**, the smallest modal logic containing $(w5)$, of which **Diff** is a normal extension.

†Segerberg refers to this property as *alio-transitivity*.

It is worth noting here that neither $(w4)$ nor $(w5)$ are Horn-formulas. Indeed, it is a routine exercise to show that any axiomatisation of **Diff** must comprise at least one non-Horn formula[†]

Despite von Wright's logic being validated by frames other than the intended class of difference frames, it was shown by de Rijke that every frame for **Diff** — that is, every symmetric, weakly-Euclidean frame — is, nonetheless, the p-morphic image of a difference frame [26].

Like all modal logics, **Diff** is a conservative extension of classical propositional logic, and hence its decision problem is trivially coNP-hard. Furthermore, it was proved in [26] that — like **S5** — **Diff** enjoys the polysize finite model property, from which we obtain a tight coNP-complete upper complexity bound[‡]. Thus the decision problems for **Diff** and **S5** share the same computational complexity. Indeed, we may even interpret **S5** as a term-definable fragment of **Diff** under the translation $(\cdot)^\dagger : \mathcal{ML}_1 \to \mathcal{ML}_1$, given by,

$$p_j^\dagger = p_j, \qquad (\neg\psi)^\dagger = \neg\psi^\dagger, \qquad (\psi_1 \wedge \psi_2)^\dagger = \psi_1^\dagger \wedge \psi_2^\dagger, \qquad (\Diamond\psi)^\dagger = \psi^\dagger \vee \Diamond\psi^\dagger,$$

for $p_j \in \textsc{prop}$. It is straightforward to check that $\varphi \in \mathbf{S5}$ if and only if $\varphi^\dagger \in \mathbf{Diff}$, for all $\varphi \in \mathcal{ML}_1$.

However, despite sharing the same computational complexity, **Diff** is somewhat more expressive than **S5** in its ability to express the uniqueness of certain worlds, and in doing so, allows for some degree of counting. Let us define the following abbreviations:

$$\Diamond^{=0}\varphi \;:=\; \neg\Diamond^{+}\varphi \qquad \text{and} \qquad \Diamond^{=1}\varphi \;:=\; \Diamond^{+}(\varphi \wedge \Box\neg\varphi), \tag{4.2}$$

where $\Diamond^{+}\varphi := \varphi \vee \Diamond\varphi$. It is straightforward to check that, given a rooted model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ whose underlying frame $\mathfrak{F} = (W, R) \in \mathsf{Fr}\,\mathbf{Diff}$ is a frame for **Diff**, we have that

$$\mathfrak{M}, w \models \Diamond^{=k}\varphi \qquad \Longrightarrow \qquad |\{u \in W : \mathfrak{M}, u \models \varphi\}| = k, \tag{4.3}$$

for all $w \in W$ and $k = 0, 1$. Furthermore, should $\mathfrak{F}$ be a genuine difference frame, in which every world is R-irreflexive, then the converse to (4.3) also holds.

[†] For **Diff** to be Horn-axiomatisable, its class of frames would have to be closed under so-called *reduced products* [21]. However, counter-examples to this can be easily constructed.

[‡] de Rijke, in turn, cites an unpublished note of de Smit and van Emde Boas [28].

Moreover, it is a straightforward exercise to show that $\Diamond\varphi$ is itself expressible in terms of $\Diamond^{>0}\varphi$ and $\Diamond^{>1}\varphi$, where $\Diamond^{>0}\varphi := \neg\Diamond^{=0}\varphi$ and $\Diamond^{>1}\varphi := \neg\Diamond^{=0}\varphi \wedge \neg\Diamond^{=1}\varphi$. Indeed, we have that

$$\Diamond\varphi \;\leftrightarrow\; (\neg\varphi \wedge \Diamond^{>0}\varphi) \vee \Diamond^{>1}\varphi$$

is a substitution instance of a propositional tautology, once we have unpacked the definitions given in (4.2), as can be easily verified.

4.3 Classical First-order Logic as Modal Logic

4.3.1 Unimodal Logics

The *standard translation* $\pi_x : \mathcal{ML}_1 \to \mathcal{L}$ provides a means by which we may interpret unimodal formulas within first-order logic. With each propositional variable $p_j \in$ PROP, we associate a monadic predicate symbol $P_j \in$ PRED, and define $\pi(\varphi)$ inductively, by taking

$$\pi(p_j) = P_j(x), \quad \pi(\neg\psi) = \neg\pi(\psi), \quad \pi(\psi_1 \wedge \psi_2) = \pi(\psi_1) \wedge \pi(\psi_2),$$
$$\pi(\Diamond\psi) = \exists y \;\big(xRy \wedge \pi(\psi)\{y/x\}\big),$$

for all $p_j \in$ PROP, where R is an auxiliary binary predicate symbol and $\pi(\psi)\{y/x\}$ is the result of uniformly substituting all instances of the variable x in $\pi(\psi)$ for the variable y, where y is a fresh variable not occurring in $\pi(\psi)$.

Furthermore, we may associate each Kripke model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ as a first-order structure $\mathfrak{A}_{\mathfrak{M}} = (W, (\cdot)^{\mathfrak{A}_{\mathfrak{M}}})$, where W comprises the set of all possible worlds of $\mathfrak{F}$, $R^{\mathfrak{A}_{\mathfrak{M}}}$ is the accessibility relation of $\mathfrak{F}$, and $P_j^{\mathfrak{A}_{\mathfrak{M}}} = \mathfrak{V}(p_j)$, for each $p_j \in$ PRED.

It follows from a routine induction that

$$\mathfrak{M}, w \models \varphi \qquad \Longleftrightarrow \qquad \mathfrak{A}_{\mathfrak{M}} \models^h \pi(\varphi),$$

for all $\varphi \in \mathcal{ML}_1$, where $h(x) = w$. In particular, we have that $\varphi \in \mathbf{K}$ if and only if $\forall x\, \pi(\varphi)$ is a theorem of $\mathcal{L}$. Thus we may identify $\mathbf{K}$ with a decidable fragment of first-order logic.

However, it was noted by Wajsberg [132] that if we, instead, consider the modal logic **S5**, then the following variation of the standard translation maps *surjectively* onto the one-variable fragment $\mathcal{L}^1$. As above, we associate each propositional variable $p_j \in$ PROP with a monadic predicate symbol $P_j \in \textsc{pred}$, and define the translation $(\cdot)^\star : \mathcal{ML}_1 \to \mathcal{L}^1$, by taking

$$p_j^\star = P_j(x), \qquad (\neg\psi)^\star = \neg\psi^\star, \qquad (\psi_1 \wedge \psi_2)^\star = \psi_1^\star \wedge \psi_2^\star, \qquad (\Diamond\psi)^\star = \exists x\ \psi^\star,$$

for $p_j \in \textsc{prop}$. Since **S5** is characterised by the class of all universal frames, it follows from a routine induction that $\varphi \in \mathbf{S5}$ if and only if $\forall x\ \varphi^\star$ is a theorem of $\mathcal{L}$.

Similarly, we may define a further variation of the standard translation which identifies von Wright's logic **Diff** within the one-variable fragment of first-order logic with the addition of some counting quantifiers $\exists_{>m}x$, for $m = 0, 1$. As above, we associate each propositional variable $p_j \in \textsc{prop}$ with a monadic predicate symbol $P_j \in \textsc{pred}$, and define the translation $(\cdot)^\dagger : \mathcal{ML}_1 \to \mathcal{C}^1$, by taking

$$p_j^\dagger = P_j(x), \qquad (\neg\psi)^\dagger = \neg\psi^\dagger, \qquad (\psi_1 \wedge \psi_2)^\dagger = \psi_1^\dagger \wedge \psi_2^\dagger, \qquad (\Diamond\psi)^\dagger = \exists^{\neq} x\ \psi^\dagger,$$

for $p_j \in \textsc{prop}$, where

$$\exists^{\neq} x\ \varphi \ := \ \big(\neg\varphi \wedge \exists_{>0}x\ \varphi\big) \ \vee \ \exists_{>1}x\ \varphi. \tag{4.4}$$

Since **Diff** is characterised by the class of all difference frames, it follows from a routine induction that $\varphi \in \mathbf{Diff}$ if and only if $\forall x\ \varphi^\dagger$ is a theorem of $\mathcal{C}$.

4.3.2 Bimodal Logics

As noted above, the translation $(\cdot)^\star : \mathcal{ML}_1 \to \mathcal{L}^1$ is invertible, and thus we may consider **S5** to be a syntactic variant of the one-variable fragment of first-order logic. The idea that this modal approach to first-order logic could be extended beyond the one-variable fragment was suggested by Quine [99] and Kuhn [71], and fully realised by Venema [130].

A related line of enquiry was pursued by Tarski and his school, who sought to approximate first-order logic with systems having a propositional character, which motivated the algebraic treatment of first-order logic [55, 61, 62, 24, 16, 89, 6].

If we restrict our attention to only two variables, then we may extend the above variation on the standard translation to yield an embedding of the product logic $\mathbf{S5} \times \mathbf{S5}$ within the two-variable fragment of first-order logic. With each propositional variable $p_j \in \textsc{prop}$, we associate a *binary* predicate symbol $P_j \in \textsc{pred}$, and define the following variation of the standard translation $(\cdot)^\star : \mathcal{ML}_2 \to \mathcal{L}^2$, by taking

$$p_j^\star = P_j(x,y), \qquad (\neg\psi)^\star = \neg\psi^\star, \qquad (\psi_1 \wedge \psi_2)^\star = \psi_1^\star \wedge \psi_2^\star,$$
$$(\Diamond_h\psi)^\star = \exists x\ \psi^\star, \qquad (\Diamond_v\psi)^\star = \exists y\ \psi^\star,$$

for all $p_j \in$ PROP. Since $\mathbf{S5} \times \mathbf{S5}$ is characterised by the class of all *square* products of two identical universal frames, it follows from a routine induction that $\varphi \in \mathbf{S5} \times \mathbf{S5}$ if and only if $\forall x \forall y\ \varphi^\star$ is a theorem of $\mathcal{L}^2$. The coNExpTime upper bound on the decision problem for $\mathbf{S5} \times \mathbf{S5}$ then follows from the decidability of the two-variable fragment of first-order logic [52].

It should be noted that, unlike with the one-variable case described above, this translation is not surjective, but instead maps surjectively onto the two-variable (equality-free), *substitution-free* fragment of first-order logic [38]. That is to say that formulas of the form $P_j(y,x)$ do not appear in the image of π. The full realisation of the two-variable fragment within modal logic, given by Venema [130], involves additional modal operators to emulate equality and transposition of variables. We shall discuss the computational complexity of products of modal logics equipped with an additional equality operator at length in Chapter 10.

As above, a similar translation embeds the product logic $\mathbf{Diff} \times \mathbf{Diff}$ within a fragment of the two-variable fragment of first-order logic *with counting quantifiers*. One caveat is that while $\mathbf{S5} \times \mathbf{S5}$ is characterised by its square product frames, $\mathbf{Diff} \times \mathbf{Diff}$ enjoys no such characterisation by its square product frames. For example, the formula $(\Box_h\bot \to \Box_v\bot)$ is valid in any square product frame for $\mathbf{Diff} \times \mathbf{Diff}$, but can be easily refuted in non-square product frames such as $(\{a\}, \neq) \times (\{a,b\}, \neq)$.

Therefore, let us denote by $\mathbf{Diff} \times^{sq} \mathbf{Diff}$, the logic characterised by all square product frames for $\mathbf{Diff} \times \mathbf{Diff}$; that is to say,

$$\mathbf{Diff} \times^{sq} \mathbf{Diff} \ := \ \mathsf{Log}\{\mathfrak{F} \times \mathfrak{F} : \mathfrak{F} \in \mathsf{Fr}\,\mathbf{Diff}\}.$$

This discrepancy between $\mathbf{Diff} \times \mathbf{Diff}$ and $\mathbf{Diff} \times^{sq} \mathbf{Diff}$, means that, while embedding $\mathbf{Diff} \times^{sq} \mathbf{Diff}$ within the two-variable fragment with counting quantifiers is relatively trivial, we must take greater care when extending the standard translation to $\mathbf{Diff} \times \mathbf{Diff}$, whose models need not be square.

As above, we associate each propositional variable $p_j \in$ PROP with a binary predicate symbol $P_j \in$ PRED. Additionally, let D_h and D_v be two monadic predicate symbols, with which we shall specify the horizontal and vertical domains, respectively. We define a new translation $(\cdot)^\dagger : \mathcal{ML}_2 \to \mathcal{C}^2$, by taking

$$p_j^\dagger = P_j(x,y), \qquad (\neg\psi)^\dagger = \neg\psi^\dagger, \qquad (\psi_1 \wedge \psi_2)^\dagger = \psi_1^\dagger \wedge \psi_2^\dagger,$$
$$(\Diamond_h\psi)^\dagger = \exists^{\neq} x\ (D_h(x) \wedge \psi^\dagger), \qquad (\Diamond_v\psi)^\dagger = \exists^{\neq} y\ (D_v(y) \wedge \psi^\dagger),$$

for $p_j \in$ PROP, where the quantifiers $\exists^{\neq} x$ and $\exists^{\neq} y$ are as defined in (4.4).

It then follows from a routine induction that $\varphi \in \mathbf{Diff} \times \mathbf{Diff}$ (resp. $\varphi \in \mathbf{Diff} \times^{sq} \mathbf{Diff}$) if and only if $\forall x \forall y\ \varphi^\dagger$ (resp. $\forall x(D_h(x) \leftrightarrow D_v(x)) \to \forall x \forall y\ \varphi^\dagger$) is a theorem of $\mathcal{C}^2$. Since the validity problem for two-variable fragment of first-order logic with counting quantifiers from among $\{\exists_{>m} x, \exists_{>m} y : m = 0, 1\}$ is decidable in coNExpTime [94], so too must be the decision problems for both $\mathbf{Diff} \times \mathbf{Diff}$ and $\mathbf{Diff} \times^{sq} \mathbf{Diff}$.

Moreover, by extending the above reduction from $\mathbf{S5}$ to $\mathbf{Diff}$, we find that $\mathbf{S5} \times \mathbf{S5}$ is term-definable within $\mathbf{S5} \times \mathbf{Diff}$, which is itself term-definable both within $\mathbf{Diff} \times \mathbf{Diff}$ and — since $\mathbf{S5} \times \mathbf{Diff}$ is also characterised by its square product frames — within $\mathbf{Diff} \times^{sq} \mathbf{Diff}$. Hence, it follows that the decision problems for each of the logics $\mathbf{Diff} \times \mathbf{Diff}$, $\mathbf{Diff} \times^{sq} \mathbf{Diff}$ and $\mathbf{S5} \times \mathbf{Diff}$ are coNExpTime-hard, akin to that of $\mathbf{S5} \times \mathbf{S5}$.

Theorem 4.3. *Let L be any of the logics* $\mathbf{Diff} \times \mathbf{Diff}$, $\mathbf{Diff} \times^{sq} \mathbf{Diff}$ *and* $\mathbf{S5} \times \mathbf{Diff}$. *Then the decision problem for L is* coNExpTime*-complete.*

Part II

Products of Modal Logics with Difference Relations

Chapter 5

Axiomatisation of Products

In this chapter we consider problems relating to the axiomatisation and finite frame problems of product logics of the form $L \times \mathbf{Diff}$, which fall outside the remit of Theorem 3.10. In Section 5.1 we show that no logic between $\mathbf{K} \times \mathbf{wK5}$ and $\mathbf{S5} \times \mathbf{Diff}$ cannot be axiomatised using only finitely many propositional variables. Furthermore, in Section 5.2 we show that the logic $\mathbf{Diff} \times^{sq} \mathbf{Diff}$, characterised by only those square product frames for $\mathbf{Diff} \times \mathbf{Diff}$, cannot be finitely axiomatised over its sublogic $\mathbf{Diff} \times \mathbf{Diff}$ — itself, not finitely axiomatisable.

Falling outside the aforementioned interval is the logic $\mathbf{Alt} \times \mathbf{Diff}$, characterised by those product frames in which the horizontal component describes a partial function. In Section 5.3 we show that $\mathbf{Alt} \times \mathbf{Diff}$ is actually finitely axiomatisable. In particular, we describe a hitherto unknown class of product matching logics, proving that $\mathbf{Alt}$ and L are product matching whenever L is canonical.

In Section 5.4, we provide a full classification of the finite frames for $\mathbf{S5} \times \mathbf{Diff}$, thereby providing a polynomial time algorithm for deciding its finite frame problem, despite the lack of any finite axiomatisation. In Section 5.5, we discuss how this classification theorem may be generalised to describe the finite frames for $\mathbf{Diff} \times \mathbf{Diff}$.

5.1 Non-finitely Axiomatisable Products

As described above in Theorem 3.10, every pair of logics, axiomatisable by some finite set of Horn formulas, are product matching, and hence their products are finitely axiomatisable. However, the situation is far less amicable for non-Horn-axiomatisable logics. In [38, Theorem 5.15], it is shown that $\mathbf{K4.3}$ and L are not product matching, whenever L is a

Kripke complete extension of **K4**, having a two-element reflexive chain among its frames.

Furthermore, in [77] it is proved that $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{K4.3})$ — far from being product matching — cannot be axiomatised using only finitely many propositional variables, whenever $\mathcal{C}$ contains an ω-fan[†]. Notable examples of such non-finitely axiomatisable product logics include $\mathbf{K} \times \mathbf{K4.3}$, $\mathbf{K4} \times \mathbf{K4.3}$ and $\mathbf{GL} \times \mathbf{K4.3}$. However it remains open whether either of the logics $\mathbf{K4.3} \times \mathbf{K4.3}$ and $\mathbf{S5} \times \mathbf{K4.3}$ can be finitely axiomatised.

It is a routine exercise to show that — like **K4.3** — **Diff** cannot be axiomatised using only Horn-formulas[‡], and so it follows that products of the form $L \times \mathbf{Diff}$ — like those of the form $L \times \mathbf{K4.3}$ — fall outside the remit of Theorem 3.10. In this section we introduce a wide interval of bimodal logics, extending $\mathbf{K} \times \mathbf{wK5}$, that do not admit any axiomatisation using only finitely many variables. Thus, we see that, despite the structural similarities shared with logics of the form $L \times \mathbf{S5}$, products such as $L \times \mathbf{Diff}$ more closely resemble those of the form $L \times \mathbf{K4.3}$ with respect to axiomatisability.

Theorem 5.1. *Let L be any bimodal logic such that*

- $\mathbf{K} \times \mathbf{wK5} \subseteq L$,
- $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$ *is a frame for L.*

Then L cannot be axiomatised using only finitely many variables.

Let L be as described, and consider the following infinite family of frames $\mathfrak{F}_k = (U_k, S_h^k, S_v^k)$, for $1 < k < \omega$, given by,

$$\begin{aligned} U_k &= \{a_0, \ldots, a_{k-1}\} \cup \{b_0, \ldots, b_{k-2}\}, \\ S_h^k &= U_k \times U_k, \\ S_v^k &= \{(a_i, a_j) : i, j < k,\ i \neq j\} \cup \{(b_i, b_j) : i, j < k-1,\ i \neq j\}. \end{aligned}$$

That is to say that $\mathfrak{F}_k$ comprises two horizontally adjoining clusters of cardinalities k and $(k-1)$, respectively, in which each element is R_v^k-irreflexive; as depicted below in Figure 5.1.

[†] A frame $\mathfrak{F} = (\omega, R)$ is said to be an *ω-fan* if $\{(0, n) : 0 < n < \omega\} \subseteq R$.

[‡] The class of frames for every Horn-axiomatisable modal logic is closed under *'reduced products'* [21], while is it easily verified that $\mathsf{Fr}\,\mathbf{Diff}$ is not.

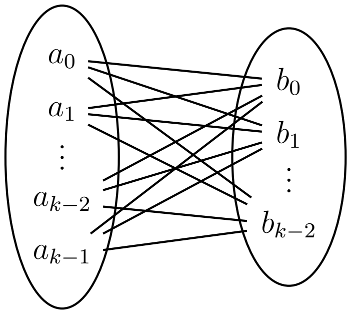


Figure 5.1: Graph representing the frame $\mathfrak{F}_k$.

In the following lemma we show that none of the frames $\mathfrak{F}_k$, for $1 < k < \omega$, validate all the theorems of $\mathbf{K} \times \mathbf{wK5}$, and *ipso facto*, cannot be a frame for L.

Lemma 5.2. *$\mathfrak{F}_k$ is not a frame for $\mathbf{K} \times \mathbf{wK5}$, for any $1 < k < \omega$.*

Proof. Suppose that $\mathfrak{F}_k$ is a frame for $\mathbf{K}\times\mathbf{wK5}$, for some $1 < k < \omega$. Then by Theorem 2.5, we have that $\mathfrak{F}_k$ is the p-morphic image of a product frame $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathsf{Fr}\,\mathbf{K}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{wK5}$. Indeed, suppose that $f : W_h \times W_v \to U_k$ is some such p-morphism. Since f is surjective, let $x_0 \in W_h$ and $y_0 \in W_v$ be such that $f(x_0, y_0) = a_0$. Since $a_0 S_v^k a_i$, for all $0 < i < k$, there must be $y_1, \ldots, y_{k-1} \in W_v$ such that $y_0 R_v y_i$ and $f(x_0, y_i) = a_i$, for all $0 < i < k$.

Since $a_0 S_h^k b_0$, there must be some $x_1 \in W_h$ such that $x_0 R_h x_1$ and $f(x_1, y_0) = b_0$. Moreover, we must have that $f(x_1, y_i) \in \{b_1, \ldots, b_{k-2}\}$ for all $0 < i < k$. Hence by the pigeon-hole principle, there must be some $i, j < k$ such that $f(x_1, y_i) = f(x_1, y_j) = b_\ell$, for some $0 < \ell < k-1$. By the supposed weak-Euclideanness of R_v, we have that either $y_i = y_j$ or $y_i R_v y_j$. However, since $f(x_0, y_i) = a_i \neq a_j = f(x_0, y_j)$ it must be that $y_i R_v y_j$, and thus $b_\ell S_v^k b_\ell$, contrary to the definition of $\mathfrak{F}_k$.

Hence, contrary to our supposition, $\mathfrak{F}_k$ is not a frame for $\mathbf{K} \times \mathbf{wK5}$, as required. □

Next we show that for sufficiently large $k < \omega$, $\mathfrak{F}_k$ is indistinguishable from a frame for L using only finitely many variables. To facilitate this, we first consider the family of frames $\mathfrak{G}_k = (V_k, T_h^k, T_v^k)$, for $0 < k < \omega$, depicted in Figure 5.2, given by,

$$
\begin{aligned}
V_k &= \{c_0, \ldots, c_{k-1}\} \cup \{d_0, \ldots, d_{k-1}\}, \\
T_h^k &= V_k \times V_k, \\
T_v^k &= \{(c_i, c_j), (d_i, d_j) : i, j \leq k \text{ and } i \neq j\} \cup \{(c_{k-1}, c_{k-1}), (d_{k-1}, d_{k-1})\}.
\end{aligned}
$$

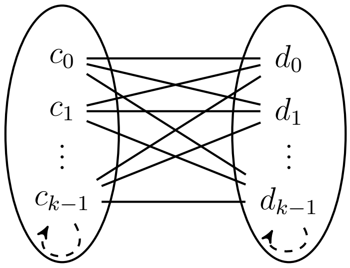


Figure 5.2: Graph representing the frame $\mathfrak{G}_k$.

Lemma 5.3. *$\mathfrak{G}_k$ is a frame for L, for all $1 < k < \omega$.*

Proof. We construct a p-morphism from $\mathfrak{H}$ onto $\mathfrak{G}_k$, where $\mathfrak{H} = (\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq) \in \mathsf{Fr}\, L$. We first define a function $\zeta_c : \mathbb{Z} \times \mathbb{Z} \to \{c_0, \ldots, c_{k-1}\}$, by taking

$$\zeta_c(i,j) = \begin{cases} c_{(i-j)} & \text{if } 0 \leq (i-j) < k-1, \\ c_{k-1} & \text{otherwise.} \end{cases}$$

Note that each c_i occurs exactly once in each row and each column of the $\mathbb{Z} \times \mathbb{Z}$ grid, except for c_{k-1} which occurs infinitely often throughout. We define $\zeta_d : \mathbb{Z} \times \mathbb{Z} \to \{d_0, \ldots, d_{k-1}\}$ analogously, with each d_i occurring exactly once in each row and each column, except for d_{k-1} which occurs infinitely often throughout.

We can then 'splice' together these two functions, taking $f : \mathbb{Z} \times \mathbb{Z} \to V_k$ to be the function given by,

$$f(n,m) = \begin{cases} \zeta_c\big(\frac{1}{2}n, m\big) & \text{if } n \text{ is even,} \\ \zeta_d\big(\frac{1}{2}(n+1), m\big) & \text{if } n \text{ is odd.} \end{cases}$$

It is then straightforward to check that f is a p-morphism of $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$ onto $\mathfrak{G}_k$. Furthermore, since $\mathfrak{H}$ is a frame for L, it follows from Proposition 2.4 that $\mathfrak{G}_k$ is also a frame for L, as required. $\square$

We may now show that $m < \omega$ propositional variables are insufficient to distinguish $\mathfrak{F}_k$ from a genuine L-frame, for sufficiently large k. We say that a model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ is *m-generated* if $\mathfrak{V}(p_i) = \emptyset$, for all $i \geq m$.

Lemma 5.4. *Let $\mathfrak{M}_m$ be an arbitrary m-generated model over $\mathfrak{F}_k$, for some $k > 2^{m+1}$. Then $\mathfrak{M}_m$ is a model for L.*

Proof. Let $k > 2^{m+1}$ and suppose that $\mathfrak{M}_m = (\mathfrak{F}_k, \mathfrak{V})$ is an m-generated model over $\mathfrak{F}_k$.

We define two equivalence relations, $\sim_a$ on $\{0, \dots, k-1\}$ and $\sim_b$ on $\{0, \dots, k-2\}$, by taking,

$$i \sim_a j \qquad \Longleftrightarrow \qquad a_i \in \mathfrak{V}(p_\ell) \text{ if and only if } a_j \in \mathfrak{V}(p_\ell) \quad \text{for all } \ell < m,$$
$$i \sim_b j \qquad \Longleftrightarrow \qquad b_i \in \mathfrak{V}(p_\ell) \text{ if and only if } b_j \in \mathfrak{V}(p_\ell) \quad \text{for all } \ell < m.$$

Since there can be at most 2^m possible $\sim_a$-equivalence classes and at most 2^m possible $\sim_b$-equivalence classes, by the pigeon-hole principle, there are $i_0 < i_1 < i_2 < k$ and $j_0 < j_1 < k-1$ such that $i_0 \sim_a i_1 \sim_a i_2$ and $j_0 \sim_b j_1$. Without any loss of generality, we may suppose that $i_0, i_1, i_2, j_0, j_1 \geq k-3$, subject to any necessary relabelling.

We may then define the following function $f : U_k \to V_{k-1}$, by taking

$$f(a_i) = \begin{cases} c_i & \text{if } i < k-3, \\ c_{k-3} & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f(b_j) = \begin{cases} d_j & \text{if } j < k-3, \\ d_{k-3} & \text{otherwise,} \end{cases}$$

for $i < k$ and $j < k-1$. It is straightforward to check that f is a p-morphism from $\mathfrak{F}_k$ onto $\mathfrak{G}_{k-1}$.

Furthermore, we may define a new model $\mathfrak{M}' = (\mathfrak{G}_{k-1}, \mathfrak{V}')$ over $\mathfrak{G}_{k-1}$, by taking, for all $u \in G_{k-1}$ and all $\ell < m$,

$$u \in \mathfrak{V}'(p_\ell) \qquad \Longleftrightarrow \qquad u = f(v) \text{ for some } v \in \mathfrak{V}(p_\ell),$$

and $\mathfrak{V}'(p_\ell) = \emptyset$ for $\ell \geq m$. By construction we have ensured that f is a model p-morphism from $\mathfrak{M}_m$ onto $\mathfrak{M}'$.

Now suppose that $\varphi \in L$. By Lemma 5.3, we have that $\mathfrak{G}_{k-1}$ is a frame for L and so $\mathfrak{M}' \models \varphi$. It then follows from Proposition 2.4 that $\mathfrak{M}_m \models \varphi$. Hence we have that $\mathfrak{M}_m$ is a model for L, as required. □

We are now in a position to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. Suppose that L is as described and that Γ constitutes an axiomatisation for L containing only finitely many variables $p_0, \dots, p_{m-1}$, for some $m < \omega$. Take $k > 2^{m+1}$ and consider an arbitrary model $\mathfrak{M} = (\mathfrak{F}_k, \mathfrak{V})$ over $\mathfrak{F}_k$. Let $\mathfrak{M}_m = (\mathfrak{F}_k, \mathfrak{V}_m)$ be the m-generated model given by the valuation

$$\mathfrak{V}_m(p_i) = \begin{cases} \mathfrak{V}(p_i) & \text{if } i < m, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is not hard to see that $\mathfrak{M} \models \Gamma$ if and only if $\mathfrak{M}_m \models \Gamma$, since $\mathfrak{M}$ and $\mathfrak{M}_m$ agree on the valuation of all propositional variables occurring in Γ.

By Lemma 5.4, we have that $\mathfrak{M}_m \models L$, and in particular, we have that $\mathfrak{M}_m \models \Gamma$, since $\Gamma \subseteq L$. Consequently $\mathfrak{M} \models \Gamma$, and since $\mathfrak{V}$ was chosen arbitrarily, it follows that $\mathfrak{F}_k \models \Gamma$.

By definition, we have that $\Gamma \subseteq \mathsf{Log}(\mathfrak{F}_k)$, and hence $L \subseteq \mathsf{Log}(\mathfrak{F}_k)$, since $\mathsf{Log}(\mathfrak{F}_k)$ is deductively closed. This is to say that $\mathfrak{F}_k$ is a frame for L. However, given that $\mathbf{K} \times \mathbf{wK5} \subseteq L$, we must have that $\mathfrak{F}_k$ is a frame for $\mathbf{K} \times \mathbf{wK5}$ contrary to Lemma 5.2.

Hence, we are then forced to conclude that any axiomatisation for L must necessarily contain infinitely many distinct propositional variables. □

In particular, if $\mathbf{K} \times \mathbf{wK5} \subseteq L \subseteq \mathbf{S5} \times \mathbf{Diff}$ that L cannot be axiomatised using only finitely many variables, since $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$ is a frame for $\mathbf{S5} \times \mathbf{Diff}$.

Corollary 5.5. *Let L be any of the logics* **K**, **T**, **K4**, **S4**, **K4.3**, **S4.3**, **Diff**, **S5**. *Then neither $L \times$* **wK5** *nor $L \times$* **Diff** *are finitely axiomatisable.*

Some notable cases, to which this result does not extend, include $\mathsf{Log}(\omega, <) \times \mathbf{Diff}$, $\mathbf{GL} \times \mathbf{Diff}$ and $\mathbf{Grz} \times \mathbf{Diff}$, which do not admit $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$ among their frames.

Question 5.6. Are any of the logics $\mathsf{Log}(\omega, <) \times \mathbf{Diff}$, $\mathbf{GL} \times \mathbf{Diff}$ and $\mathbf{Grz} \times \mathbf{Diff}$ finitely axiomatisable?

5.2 Non-finitely Axiomatisable Squares

Recall that **Diff** $\times^{sq}$ **Diff**, characterised by the class of all square product frames for **Diff** $\times$ **Diff**, is a strict extension of **Diff** $\times$ **Diff**. Since, **Diff** $\times^{sq}$ **Diff** falls within the remit of Theorem 5.1, it follows that **Diff** $\times^{sq}$ **Diff** cannot be axiomatised using only finitely many variables. However, due to the similar structure of their characteristic frames, it is reasonable to ask whether it can be finitely axiomatised over **Diff** $\times$ **Diff**.

Contrary to this suggestion, we show here that no logic extending **Diff** $\times^{sq}$ **Diff** and having $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$ among its frames can be axiomatised over **Diff** $\times$ **Diff** using only finitely many variables.

Theorem 5.7. *Let L be any bimodal logic such that*

- **Diff** $\times^{sq}$ **Diff** $\subseteq L$,
- $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$ *is a frame for L.*

Then L cannot be axiomatised over **Diff** $\times$ **Diff** *using only finitely many variables.*

It is worth noting here that, despite appearances, $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$ is indeed a frame for **Diff** $\times^{sq}$ **Diff**; being as it is, a p-morphic image of $(\mathbb{Z}, \neq) \times (\mathbb{Z}, \neq)$. Consider, for example, the function $f : \mathbb{Z}^2 \to \mathbb{Z}^2$, given by

$$f(n,m) = \begin{cases} \left(\frac{1}{2}n, m\right) & \text{if } n \text{ is even,} \\ \left(\frac{1}{2}(n+1), m\right) & \text{if } n \text{ is odd,} \end{cases}$$

for all $n, m \in \mathbb{Z}$. It is straightforward to check that f is a p-morphism from $(\mathbb{Z}, \neq) \times (\mathbb{Z}, \neq)$ onto $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$. Thus, Theorem 5.7 describes a non-empty interval of logics extending **Diff** $\times^{sq}$ **Diff**.

We describe new family of frames $\widetilde{\mathfrak{F}}_k = (\widetilde{U}_k, \widetilde{S}^k_h, \widetilde{S}^k_v)$, for $1 < k < \omega$, akin those described above, by taking,

$$\begin{aligned} \widetilde{U}_k &= \{a_0, \dots, a_{k-1}\} \cup \{b_0, \dots, b_{k-1}\}, \\ \widetilde{S}^k_h &= \widetilde{U}_k \times \widetilde{U}_k, \\ \widetilde{S}^k_v &= \{(a_i, a_j) : i, j < k,\ i \neq j\} \cup \{(b_i, b_j) : i, j < k,\ i \neq j\}. \end{aligned}$$

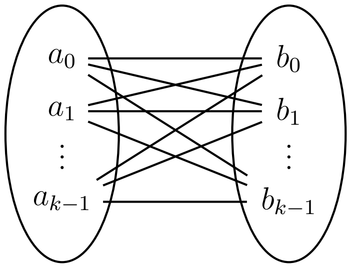


Figure 5.3: Graph representing the frame $\widetilde{\mathfrak{F}}_k$.

Lemma 5.8. *$\widetilde{\mathfrak{F}}_k$ is a frame for* $\mathbf{S5} \times \mathbf{Diff}$*, for all* $1 < k < \omega$*.*

Proof. Let $\mathfrak{H}_h = (2k, 2k \times 2k) \in \mathsf{Fr}\,\mathbf{S5}$ and $\mathfrak{H}_v = (k, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$. We define a function $f : 2k \times k \to \widetilde{U}_k$, by taking

$$f(i,j) = \begin{cases} a_\ell & \text{if } i < k \text{ and } \ell = i + j \mod k, \\ b_\ell & \text{if } i \geq k \text{ and } \ell = i + j \mod k. \end{cases}$$

It is straightforward to check that f is a p-morphism from $\mathfrak{H}_h \times \mathfrak{H}_v$ onto $\widetilde{\mathfrak{F}}_k$. Furthermore, since $\mathfrak{H}_h \times \mathfrak{H}_v$ is a frame for $\mathbf{S5} \times \mathbf{Diff}$, it follows from Proposition 2.4 that $\widetilde{\mathfrak{F}}_k$ is also a frame for $\mathbf{S5} \times \mathbf{Diff}$, as required. $\square$

It follows that $\widetilde{\mathfrak{F}}_k$ is also a frame for $\mathbf{Diff} \times \mathbf{Diff}$, for all $1 < k < \omega$, since $\mathbf{S5} \times \mathbf{Diff}$ is a normal extension of $\mathbf{Diff} \times \mathbf{Diff}$. However, we now show that $\widetilde{\mathfrak{F}}_k$, does not validate all the theorems of $\mathbf{Diff} \times^{sq} \mathbf{Diff}$, and *ipso facto*, cannot be a frame for L.

Lemma 5.9. *$\widetilde{\mathfrak{F}}_k$ is not a frame for* $\mathbf{Diff} \times^{sq} \mathbf{Diff}$*, for any* $1 < k < \omega$*.*

Proof. Suppose that $\widetilde{\mathfrak{F}}_k$ is a frame for $\mathbf{Diff} \times^{sq} \mathbf{Diff}$, for some $1 < k < \omega$. Then by Theorem 2.5, we have that $\widetilde{\mathfrak{F}}_k$ is the p-morphic image of some square product frame $\mathfrak{F} \times \mathfrak{F}$, where $\mathfrak{F} = (W, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$. Indeed, suppose that $f : W \times W \to \widetilde{U}_k$ is some such p-morphism. It follows that there are $r, w_0, \ldots, w_{2k-1} \in W$ such that $w_i \neq w_j$, for all $i \neq j$,

$$f(w_i, r) = \begin{cases} a_i & \text{if } i < k, \\ b_{i-k} & \text{if } i \geq k, \end{cases}$$

for all $i < k$.

Moreover, since f is a p-morphism, we must have that $f(r, w_i) \in \{a_1, \ldots, a_{k-1}\}$ for all $i < 2k$. Hence by the pigeon-hole principle, there must be some $i, j < 2k$ such that $f(r, w_i) = f(r, w_j) = a_\ell$, for some $\ell < k$. This is to say that $b_\ell \widetilde{S}_v^k b_\ell$, contrary to the definition of $\widetilde{\mathfrak{F}}_k$. Hence, $\widetilde{\mathfrak{F}}_k$ is not a frame for $\mathbf{Diff} \times^{sq} \mathbf{Diff}$, as required. $\square$

We take $\mathfrak{G}_k$, as defined above, and show that, for sufficiently few propositional variables, it is impossible to distinguish $\widetilde{\mathfrak{F}}_k$ from a genuine frame for $\mathbf{Diff} \times^{sq} \mathbf{Diff}$.

Lemma 5.10. *Let $\mathfrak{M}_m$ be an arbitrary m-generated model over $\widetilde{\mathfrak{F}}_k$, for some $k > 2^m$. Then $\mathfrak{M}_m$ is a model for L.*

Proof. Analogous to that of Lemma 5.4 above, however here we require only that $k > 2^m$, since we need only invoke the pigeon-hole principle to find a single 'overlap' in each cluster. $\square$

With this, we are now in a position to show that no axiomatisation containing only finitely many variables, is sufficient to describe L relative to $\mathbf{Diff} \times \mathbf{Diff}$.

Proof of Theorem 5.7. Suppose that L is as described and that Γ constitutes an axiomatisation for L over $\mathbf{Diff} \times \mathbf{Diff}$ containing only finitely many variables $p_0, \ldots, p_{m-1}$, for some $m < \omega$. Take $k > 2^m$ and consider an arbitrary model $\mathfrak{M} = (\widetilde{\mathfrak{F}}_k, \mathfrak{V})$ over $\widetilde{\mathfrak{F}}_k$. Let $\mathfrak{M}_m = (\widetilde{\mathfrak{F}}_k, \mathfrak{V}_m)$ be the m-generated model given by the valuation

$$\mathfrak{V}_m(p_i) = \begin{cases} \mathfrak{V}(p_i) & \text{if } i \leq m, \\ \emptyset & \text{otherwise.} \end{cases}$$

Again, we have that $\mathfrak{M} \models \Gamma$ if and only if $\mathfrak{M}_m \models \Gamma$, since $\mathfrak{M}$ and $\mathfrak{M}_m$ agree on the valuation of all propositional variables occurring in Γ.

By Lemma 5.10, we have that $\mathfrak{M}_m \models L$, and in particular, we have that $\mathfrak{M}_m \models \Gamma$, since $\Gamma \subseteq L$. Consequently $\mathfrak{M} \models \Gamma$, and since $\mathfrak{V}$ was chosen arbitrarily, it follows that $\mathfrak{F}_k \models \Gamma$.

Thus, by definition, we have that $\Gamma \subseteq \mathsf{Log}(\widetilde{\mathfrak{F}}_k)$. Moreover, it follows from Lemma 5.8 that $\widetilde{\mathfrak{F}}_k$ is a frame for $\mathbf{Diff} \times \mathbf{Diff}$, and thus $\mathbf{Diff} \times \mathbf{Diff} \cup \Gamma \subseteq \mathsf{Log}(\widetilde{\mathfrak{F}}_k)$. Hence, it follows that $\mathbf{Diff} \times^{sq} \mathbf{Diff} \subseteq \mathsf{Log}(\widetilde{\mathfrak{F}}_k)$, since $\mathsf{Log}(\widetilde{\mathfrak{F}}_k)$ is deductively closed. This is to say that $\widetilde{\mathfrak{F}}_k$ is a frame for $\mathbf{Diff} \times^{sq} \mathbf{Diff}$, contrary to Lemma 5.9.

Hence, it follows that L cannot be finitely axiomatised over $\mathbf{Diff} \times \mathbf{Diff}$, as required. $\square$

Thus we see that, not only can **Diff** $\times^{sq}$ **Diff** not be axiomatised using only finitely many variables, but it cannot even be finitely axiomatised over **Diff** $\times$ **Diff**, which itself cannot be finitely axiomatised.

Question 5.11. Is it possible to give a description of an infinite family of formulas that, together, are sufficient to axiomatise **Diff** $\times^{sq}$ **Diff** over **Diff** $\times$ **Diff**?

5.3 Finitely Axiomatisable Products

Other notable cases that lie outside the remit of Theorem 5.1 are the logics **Alt** $\times$ **Diff**, **Alt** $\times$ **wK5** and **Alt** $\times$ **K4.3**, where **Alt** $:= \mathbf{K} + (\Diamond p \to \Box p)$ is the logic characterised by all those frames whose accessibility relation describes a partial function; that is to say, in which every world has at most one R-successor.

While n-dimensional products of the form **Alt** $\times \cdots \times$ **Alt** were considered and shown to be finitely axiomatisable by their respective n-ary commutators $[\mathbf{Alt}, \ldots, \mathbf{Alt}]$ in [39], there seems to have been little interest in their products outside of the typically considered Horn-axiomatisable logics.

Here we show that, not only are the logics **Alt** $\times$ **Diff**, **Alt** $\times$ **wK5** and **Alt** $\times$ **K4.3** all finitely axiomatisable, but that **Alt** and L are product matching, whenever L is canonical. The result rests upon the following lemma.

Lemma 5.12. *Let $\mathfrak{G} = (W, R_h, R_v)$ be a countable, rooted frame for $[\mathbf{Alt}, L]$. Then $\mathfrak{G}$ is the p-morphic image of a product frame $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_h \in \mathsf{Fr}\,\mathbf{Alt}$ and $\mathfrak{F}_v \in \mathsf{Fr}\,L$.*

Proof. Let $\mathfrak{G}$ be as defined and let $r \in W$ be any root for $\mathfrak{G}$. We define a function $\rho : W \to W \cup \{\bot\}$ such that $\rho(x) = y$ if and only if xR_hy and $\rho(x) = \bot$ if x has no R_h-successors. Since $\mathfrak{G}$ is a frame for $[\mathbf{Alt}, L]$, it follows that ρ is well-defined.

We define $\mathfrak{F}_h = (W_h, S_h)$ and $\mathfrak{F}_v = (W_v, S_v)$, by taking,

$$W_h = \{k < \omega : \rho^k(r) \neq \bot\} \qquad \text{and} \qquad W_v = \{w \in W : rR_v^k w \text{ for some } k < \omega\},$$

where S_h is the successor relation on $W_h \in 2^\omega$, and S_v is the restriction of R_v to W_v.

Now take $f : W_h \times W_v \to W$ to be the function given by,

$$f(k, y) = \rho^k(y),$$

for all $k \in W_h$ and $y \in W_v$. The surjectivity of f follows from the commutativity of R_h and R_v, and $\mathfrak{G}$ being rooted at $r \in W$. It remains to show that f is a p-morphism of $\mathfrak{F}_h \times \mathfrak{F}_v$ onto $\mathfrak{G}$.

– Suppose that xS_hx' then by definition $x = k$ and $x' = k + 1$, for some $k < \omega$. Hence we have that $f(x, y) = \rho^k(y)$ and $f(x', y) = \rho^{k+1}(y) = \rho\big(\rho^k(y)\big) = \rho\big(f(x, y)\big)$, which is to say that $f(x, y)R_hf(x', y)$.

– Suppose that $f(k, y)R_hw$ for some $w \in W$. That is to say that $\rho^{k+1}(y) = \rho(f(k, y)) = w \neq \perp$. It then follows from (com^l) that $\rho^{k+1}(r) \neq \perp$, and so $(k+1) \in W_h$. Hence there is some $(k + 1) \in W_h$ such that $f(k + 1, y) = w$.

– Suppose that yS_vy' then by definition yR_vy', which is to say that $f(0, y)R_vf(0, y')$. Now suppose that $f(k, y)R_vf(k, y')$, for some $k \in W_h$ such that $(k + 1) \in W_h$. By definition we have that $f(k, y)R_hf(k + 1, y)$ and so it follows from (chr) that there is some $u \in W$ such that $f(k, y')R_hu$ and $f(k + 1, y)R_vu$. However by definition we have that $u = f(k + 1, y')$, and so $f(k + 1, y)R_vf(k + 1, y')$. Hence by induction we have that $f(x, y)R_vf(x, y')$ for all $x \in W_h$.

– Suppose that $f(k, y)R_vw$ for some $w \in W$. It follows from (com^r) that there is some $y' \in W_v$ such that yR_vy' and $f(k, y') = w$.

Hence we conclude that $\mathfrak{G}$ is the p-morphic image of a product frame $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_h \in \mathsf{Fr}\,\mathbf{Alt}$ and $\mathfrak{F}_v \in \mathsf{Fr}\,L$, as required. □

It is then straightforward to show that **Alt** and L are product matching, whenever L is canonical.

Theorem 5.13. *Let L be any canonical modal logic. Then* **Alt** *and L are product matching.*

Proof. Clearly $[\mathbf{Alt}, L] \subseteq \mathbf{Alt} \times L$, as is true of all product logics. Conversely, since both **Alt**, and L are canonical, it follows from Theorem 3.8 that $[\mathbf{Alt}, L]$ too is canonical, and thus Kripke complete. Now suppose that $\varphi \notin [\mathbf{Alt}, L]$. Then $\mathfrak{M}, w \not\models \varphi$ for some model $\mathfrak{M} = (\mathfrak{G}, \mathfrak{V})$, where $\mathfrak{G}$ is a countable, rooted frame for $[\mathbf{Alt}, L]$. By Lemma 5.12, $\mathfrak{G}$ is the p-morphic image of some product frame $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_h \in \mathsf{Fr}\,\mathbf{Alt}$ and $\mathfrak{F}_h \in \mathsf{Fr}\,L$.

Suppose f is some such p-morphism, and define a new model $\mathfrak{M}' = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V}')$ by taking $(x, y) \in \mathfrak{V}'(p)$ if and only if $f(x, y) \in \mathfrak{V}(p)$, for all propositional variables $p \in \textsc{prop}$. It then follows from Proposition 2.4 that $\mathfrak{M}', w \not\models \varphi$, and so $\varphi \notin \mathbf{Alt} \times L$, as required. $\square$

As mentioned above, it follows that **Alt** and L are product matching, for each of the logics $L \in \{\mathbf{Diff}, \mathbf{wK5}, \mathbf{K4.3}\}$. Hence, each of the product logics $\mathbf{Alt} \times \mathbf{Diff}$, $\mathbf{Alt} \times \mathbf{wK5}$ and $\mathbf{Alt} \times \mathbf{K4.3}$ are finitely axiomatisable; a stark contrast to those logics described in Theorem 5.1.

Corollary 5.14. *Let L be any of the logics* **Diff**, **wK5**, *or* **K4.3**. *Then* $\mathbf{Alt} \times L$ *is finitely axiomatisable.*

5.4 The Finite Frame Problem

As discussed above in Section 2.1.2, it is a trivial task to decide whether an arbitrary finite frame is a frame for a modal logic L, whenever L is finitely axiomatisable; simply check that each of the finitely many axioms are valid in a given finite frame. The task need not be so straightforward for logics that are not finitely axiomatisable.

However, despite having no finite axiomatisation, the finite frame problems for $\mathbf{S5} \times \mathbf{Diff}$ and $\mathbf{Diff} \times \mathbf{Diff}$ are nonetheless decidable; a result that follows from a routine consideration of Jankov-Fine frame formulas.

Theorem 5.15. *Let $L_i \in \{\mathbf{S5}, \mathbf{Diff}\}$, for $i = h, v$. Then the finite frame problem for $L_h \times L_v$ is decidable.*

Proof. Suppose, without any loss of generality, that $\mathfrak{F} = (n, R_h, R_v)$ is an arbitrary finite bimodal frame, rooted at $0 < n$, for some $n < \omega$. With each $i < n$ we associate a propositional variable $p_i \in \textsc{prop}$, and define the *Jankov-Fine formula* for $\mathfrak{F}$ to be:

$$\chi(\mathfrak{F}) \;:=\; \Box_h^+ \Box_v^+ \delta(\mathfrak{F}) \to \neg p_0,$$

where,

$$\begin{aligned}\delta(\mathfrak{F}) \quad := \quad & \bigwedge_{iR_hj}(p_i \to \Diamond_h p_j) \;\wedge\; \bigwedge_{\neg iR_hj}(p_i \to \neg\Diamond_h p_j) \\ & \wedge \bigwedge_{iR_vj}(p_i \to \Diamond_v p_j) \;\wedge\; \bigwedge_{\neg iR_vj}(p_i \to \neg\Diamond_v p_j) \\ & \wedge \bigwedge_{i\neq j}(p_i \to \neg p_j) \;\wedge\; \bigvee_{i<n} p_i.\end{aligned}$$

It is a well-known result that $\chi(\mathfrak{F})$ is refuted in a frame $\mathfrak{G}$ if and only if $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$ [34, 33].

We claim that $\mathfrak{F}$ is a frame for $L_h \times L_v$ if and only if $\chi(\mathfrak{F}) \notin L_h \times L_v$.

($\Rightarrow$) Suppose that $\mathfrak{F}$ is a frame for $L_h \times L_v$. Since $\mathfrak{F}$ is trivially a p-morphic image of itself, it follows that $\chi(\mathfrak{F})$ is refuted in $\mathfrak{F}$, and hence $\chi(\mathfrak{F}) \notin L_h \times L_v$.

($\Leftarrow$) Conversely, suppose that $\chi(\mathfrak{F}) \notin L_h \times L_v$ then $\mathfrak{F}_h \times \mathfrak{F}_v \not\models \chi(\mathfrak{F})$, where $\mathfrak{F}_i \in \mathsf{Fr}\, L_i$, for $i = h, v$. Hence it follows that $\mathfrak{F}$ is a p-morphic image of $\mathfrak{F}_h \times \mathfrak{F}_v$. Thus, by Theorem 2.4, $\mathfrak{F}$ is a frame for $L_h \times L_v$.

Since the decision problem for $L_h \times L_v$ is decidable, whenever $L_h, L_v \in \{\mathbf{S5}, \mathbf{Diff}\}$, so too must be the finite frame problem for $L_h \times L_v$, as required. $\square$

Alternatively, it follows from a general result of [75] that **S5** $\times$ **Diff** is finitely axiomatisable over **Diff** $\times$ **Diff** with the addition of the reflexivity axiom $(T) := \Box_h p \to p$. Hence, the decidability of the finite frame problem for **S5** $\times$ **Diff** can also be derived from that of **Diff** $\times$ **Diff**; it is enough to check first that a given frame is reflexive in the horizontal component and then, in the affirmative, check whether it is also a frame for **Diff** $\times$ **Diff**.

While this argument demonstrates that the finite frame problems for both **Diff** $\times$ **Diff** and **S5** $\times$ **Diff** are decidable, it offers little insight into the structure of their finite frames. Moreover, since the decision problem for **Diff** $\times$ **Diff** is coNExpTime-complete, the above procedure is far from tractable.

In what follows, we offer a more careful analysis of the structure of the frames for **S5** $\times$ **Diff**, and improve upon this ineffectual upper bound by providing a more tractable, polynomial time, decision procedure for its finite frame problem.

Definition 5.16. Let $\mathfrak{F} = (W, R_h, R_v)$ be an arbitrary bimodal frame for [**Diff**, **Diff**] and define an equivalence relation $\sim$ on W by taking,

$$u \sim v \qquad \iff \qquad uR_h^+ v \quad \text{and} \quad uR_v^+ v,$$

for all $u, v \in W$. Let $W^\sim$ be the set of all $\sim$-equivalence classes, which will henceforth be referred to as *clusters*.

We say that $\mathfrak{F}$ is described by a *grid of clusters* if there is some tuple (X, Y, h), comprising non-empty sets X, Y and a function $h : X \times Y \to W^\sim$ such that, for all $x, x' \in X$ and $y, y' \in Y$,

(gc1) If $x \neq x'$ then $uR_h v$, for all $u \in h(x, y)$ and $v \in h(x', y)$,

(gc2) If $y \neq y'$ then $uR_v v$, for all $u \in h(x, y)$ and $v \in h(x, y')$.

In what follows, we will typically associate the subset $h(x, y)$ with the *subframe* of $\mathfrak{F}$ induced by $h(x, y)$; that is to say, the frame $\mathfrak{F}_{x,y} = (h(x, y), R'_h, R'_v)$, where $R'_i = R_i \cap \big(h(x, y) \times h(x, y)\big)$, for $i = h, v$.

The following proposition stipulates that *every* rooted frame for [**Diff**, **Diff**] can be described by a grid of clusters. This rigid structure allows us to recognise those finite frames for [**Diff**, **Diff**], and its extensions, without recourse to unilluminating reductions such as that described in Theorem 5.15. This heightened lucidity allows us identify possible frames for **S5** $\times$ **Diff** and **Diff** $\times$ **Diff**, with much greater efficiency.

Proposition 5.17. *Every rooted frame for* [**Diff**, **Diff**] *can be described by a grid of clusters.*

Proof. Suppose $\mathfrak{F} = (W, R_h, R_v)$ is a frame for [**Diff**, **Diff**] with root $r \in W$, and define $U_h, U_v \subseteq W$ by taking,

$$U_h = \{w \in W : rR_h^+ w\} \qquad \text{and} \qquad U_v = \{w \in W : rR_v^+ w\}.$$

Thus, for all $(u, v) \in U_h \times U_v$ we have that $rR_h^+ u$ and $rR_v^+ v$. Furthermore, since $\mathfrak{F}$ is a frame for the Church-Rosser axiom (*chr*), we may define a function $\widehat{h} : U_h \times U_v \to W$, by choosing $\widehat{h}(u,v)$ such that $uR_v^+\widehat{h}(u,v)$ and $vR_h^+\widehat{h}(u,v)$.

We may then define

$$X = \{[u] \in W^\sim : u \in U_h\} \qquad \text{and} \qquad Y = \{[v] \in W^\sim : v \in U_v\},$$

where $[w] \in W^\sim$ is the $\sim$-equivalence class containing $w \in W$.

Now suppose that $u, u' \in U_h$ and $v, v' \in U_v$ are such that $u \sim u'$ and $v \sim v'$. In particular, uR_v^+u' and vR_h^+v'. Furthermore, since both R_h^+ and R_v^+ are both equivalence relations, we have that $\widehat{h}(u,v)R_h^+\widehat{h}(u',v')$ and $\widehat{h}(u,v)R_v^+\widehat{h}(u',v')$. This is to say that $\widehat{h}(u,v) \sim \widehat{h}(u',v')$. Hence, we may define a new function $h : X \times Y \to W^\sim$, by taking,

$$h(x,y) = \{w \in W : w \sim \widehat{h}(u,v) \text{ for some } u \in x,\ v \in y\},$$

for all $x \in X$ and $y \in Y$. It is straightforward to check that (X, Y, h) satisfies conditions **(gc1)**–**(gc2)**, and consequently $\mathfrak{F}$ may be described by a grid of clusters, as required. $\square$

By Theorem 2.5 we have that an arbitrary frame $\mathfrak{F}$ is a frame for $\mathbf{S5} \times \mathbf{Diff}$ if and only if its ultrafilter extension $\mathfrak{Ue}\mathfrak{F}$ is the p-morphic image of some product frame $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_h = (W_h, W_h^2) \in \mathsf{Fr}\,\mathbf{S5}$ and $\mathfrak{F}_v = (W_v, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$.

The following lemma demonstrates that a any frame that can be described by a grid of clusters is the p-morphic image of a product frame only if each cluster is, itself, a p-morphic image of a product frame.

Lemma 5.18. *Suppose that f is a p-morphism from $\mathfrak{F}_h \times \mathfrak{F}_v$ onto $\mathfrak{F}$. Then for every equivalence class $h \in W^\sim$, there is some $A \subseteq W_h$ and $B \subseteq W_v$ such that*

$$(x,y) \in A \times B \qquad \Longleftrightarrow \qquad f(x,y) \in h.$$

Proof. Since f is surjective, let $(a,b) \in W_h \times W_v$ be such that $f(a,b) \in h$ and choose

$$\begin{aligned} A &= \{x \in W_h : f(x,b) \in h\}, \\ B &= \{y \in W_v : f(a,y) \in h\}. \end{aligned}$$

We show that $(x,y) \in A \times B$ if and only if $f(x,y) \in h$, for all $x \in W_h$ and $y \in W_v$.

($\Rightarrow$) Suppose that $(x, y) \in A \times B$. Then by definition $f(x, b), f(a, y) \in h$. In particular, we have that $f(a, y)R_h^+ f(a, b)$ and $f(x, b)R_v^+ f(a, b)$. Furthermore, since f is a p-morphism, we have that $f(a, y)R_h^+ f(x, y)$ and $f(x, b)R_v^+ f(x, y)$. It then follows from the weak-Euclideanness of both R_h and R_v that $f(x, y)R_h^+ f(a, b)$ and $f(x, y)R_v^+ f(a, b)$. That is to say that $f(x, y) \sim f(a, b)$, and hence $f(x, y) \in h$.

($\Leftarrow$) Conversely, suppose that $f(x, y) \in h$. Since f is a p-morphism we have that $f(x, y)R_h^+ f(a, y)$ and $f(x, y)R_v^+ f(x, b)$. Furthermore, since $f(x, y) \sim f(a, b)$, we have that $f(x, y)R_h^+ f(a, b)$ and $f(x, y)R_v^+ f(a, b)$. It then follows from the weak-Euclideanness of both R_h and R_v that $f(a, y)R_h^+ f(a, b)$ and $f(x, b)R_v^+ f(a, b)$. Moreover, since f is a p-morphism, we also have that $f(a, y)R_v^+ f(a, b)$ and $f(x, b)R_h^+ f(a, b)$. That is to say that $f(x, b) \sim f(a, b) \sim f(a, y)$, and hence $(x, y) \in A \times B$, as required.

$\square$

Consequently, it follows that $h(x, y)$ induces a p-morphism onto $\mathfrak{F}_{x,y}$, whose pre-image is some subframe of $\mathfrak{F}_h \times \mathfrak{F}_v$, that is, itself, a product frame, for all $x \in X$ and $y \in Y$.

Corollary 5.19. *If $\mathfrak{F}$ is a p-morphic image of a product frame then $\mathfrak{F}_{x,y}$ is the p-morphic image of a product frame, for all $x \in X$ and $y \in Y$.*

However, the converse fails, for it is not enough that each cluster be the p-morphic image of a product frame, as the individual dimensions of each product frame may be incompatible with each other. We introduce the following notation to make precise this notion of compatible dimensions.

Definition 5.20. With each $(x, y) \in X \times Y$, we associate a set $K(x, y) \subseteq (\omega+1) \times (\omega+1)$ such that $(n, m) \in K(x, y)$ if and only if $\mathfrak{F}_{x,y}$ is a p-morphic image of $\mathfrak{F}_h \times \mathfrak{F}_v$, where $\mathfrak{F}_h = (n, n \times n) \in \mathsf{Fr}\,\mathbf{S5}$ and $\mathfrak{F}_v = (m, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$.

That is to say that $K(x, y)$ describes the range of possible dimensions for product frames that can be p-morphically mapped onto $h(x, y)$. The following theorem now gives the necessary and sufficient conditions for when an arbitrary (countable) frame is the p-morphic image of a product frame, in terms of the constraints placed upon the individual clusters.

Theorem 5.21. *Let $\mathfrak{F}$ be a countable frame for* $[\mathbf{S5}, \mathbf{Diff}]$ *and let* (X, Y, h) *be the grid of clusters describing* $\mathfrak{F}$*. Then* $\mathfrak{F}$ *is the p-morphic image of a product frame for* $\mathbf{S5} \times \mathbf{Diff}$ *if and only if there is some* $\lambda_h : X \to (\omega + 1)$ *and* $\lambda_v : Y \to (\omega + 1)$*, such that,*

$$\big(\lambda_h(x), \lambda_v(y)\big) \in K(x, y),$$

for all $x \in X$ *and* $y \in Y$*.*

Proof. ($\Rightarrow$) Suppose that f is a p-morphic image from $\mathfrak{F}_h \times \mathfrak{F}_v$ onto $\mathfrak{F}$, where $\mathfrak{F}_h = (W_h, W_h \times W_h) \in \mathsf{Fr}\,\mathbf{S5}$ and $\mathfrak{F}_v = (W_v, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$.

By Lemma 5.18, for all $x \in X$ and $y \in Y$ there is some $A_{x,y} \subseteq W_h$ and $B_{x,y} \subseteq W_v$ such that

$$(a, b) \in A_{x,y} \times B_{x,y} \qquad \Longleftrightarrow \qquad f(a, b) \in h(x, y).$$

Moreover, we claim that $B_{x,y} = B_{x',y}$, for all $x, x' \in X$ and $y \in Y$. For suppose that $x \neq x'$ and $(a, b) \in A_{x,y} \times B_{x,y}$. Then by construction $f(a, b) \in h(x, y)$. Let $w \in h(x', y)$. Then by definition we have that $f(a, b)R_h w$. Since f is a p-morphism, there is some $a' \in W_h$ such that $f(a', b) = w \in h(x', y)$, which is to say that $(a', b) \in A_{x',y} \times B_{x',y}$. Whence we have that $B_{x,y} \subseteq B_{x',y}$, and by an analogous argument, $B_{x',y} \subseteq B_{x,y}$.

Similarly we deduce that $A_{x,y} = A_{x,y'}$, for all $x \in X$ and $y, y' \in Y$.

Thus we may unambiguously define $\lambda_h : X \to (\omega + 1)$ and $\lambda_v : Y \to (\omega + 1)$, by taking,

$$\lambda_h(x) = |A_{x,y}| \qquad \text{and} \qquad \lambda_v(y) = |B_{x,y}|\,,$$

for all $x \in X$ and $y \in Y$. It follows from the definition that $\big(\lambda_h(x), \lambda_v(y)\big) \in K(x, y)$ for all $x \in X$ and $y \in Y$, as required.

($\Leftarrow$) Suppose that there are functions $\lambda_h : X \to (\omega + 1)$ and $\lambda_v : Y \to (\omega + 1)$ such that $\big(\lambda_h(x), \lambda_v(y)\big) \in K(x, y)$ for all $x \in X$ and $y \in Y$.

By definition, we have that each $\mathfrak{F}_{x,y}$, for $x \in X$ and $y \in Y$, is the p-morphic image of the product frame $\mathfrak{H}_h \times \mathfrak{H}_v$, where $\mathfrak{H}_h = (\lambda_h(x), \lambda_h(x) \times \lambda_h(x)) \in \mathsf{Fr}\,\mathbf{S5}$ and $\mathfrak{H}_v = (\lambda_v(y), \neq) \in \mathsf{Fr}\,\mathbf{Diff}$. So suppose that $g_{x,y} : \lambda_h(x) \times \lambda_v(y) \to f(x, y)$ is some such p-morphism, for each $x \in X$ and $y \in Y$. We then define a p-morphism of the whole frame by 'stitching together' these partial p-morphisms.

We take $\mathfrak{F}_h = (\Lambda_h, \Lambda_h^2) \in \mathsf{Fr}\,\mathbf{S5}$ and $\mathfrak{F}_v = (\Lambda_v, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, where

$$\begin{aligned}\Lambda_h &= \{(x,i) : x \in X \text{ and } i < \lambda_h(x)\},\\ \Lambda_v &= \{(y,j) : y \in Y \text{ and } j < \lambda_v(y)\},\end{aligned}$$

and define the function $f : \Lambda_h \times \Lambda_v \to W$, by taking,

$$f\big((x,i),(y,j)\big) = g_{x,y}(i,j),$$

for all $(x,i) \in \Lambda_h$ and $(y,j) \in \Lambda_v$. It is straightforward to check that f is a p-morphism from $\mathfrak{F}_h \times \mathfrak{F}_v$ onto the whole frame $\mathfrak{F}$, as required.

$\square$

Thus, to determine whether a finite frame for $[\mathbf{S5}, \mathbf{Diff}]$ is a frame for $\mathbf{S5} \times \mathbf{Diff}$, it is enough to check whether there exist functions $\lambda_h : X \to (\omega+1)$ and $\lambda_v : Y \to (\omega+1)$, satisfying the above criterion.

The following theorem describes the necessary and sufficient conditions for membership of $K(x,y)$. Indeed, what is shown is that $K(x,y)$ describes a linear set of constraints whose bounds are detailed below in Table 5.1.

$h(x,y)$		$K(x,y)$
$k > 0$	$t = 0$	$\{(n,k) : n \geq k\}$
$k > 0$	$t > 0$	$\{(n,m) : n \geq m, \text{ and } m \geq k + 2t\}$
$k = 0$	$t > 0$	$\{(n,m) : n \geq t \text{ and } m \geq 2t\}$

Table 5.1: Table of linear constraints associated with $\mathbf{S5} \times \mathbf{Diff}$.

Theorem 5.22 (Classification Theorem). *Let $\mathfrak{F} = (W, R_h, R_v)$ be a finite frame for* $[\mathbf{S5}, \mathbf{Diff}]$ *and let (X, Y, h) be the grid of clusters describing $\mathfrak{F}$. Suppose that $x \in X$ and $y \in Y$, and let*

$$k = |\{w \in h(x,y) : \neg w R_v w\}| \qquad \text{and} \qquad t = |\{w \in h(x,y) : w R_v w\}|.$$

Then $(n, m) \in K(x, y)$ if and only if the following conditions hold:

(i) *$n \geq k + t$ and $m \geq k + 2t$,*

(ii) *If $t = 0$ then $m \leq k$,*

(iii) *If $k > 0$ then $n \geq m$.*

Proof. (⇒) Suppose that $(n, m) \in K(x, y)$ and let $f : n \times m \to h(x, y)$ be a p-morphism from $(n, n^2) \times (m, \neq)$ onto $\mathfrak{F}_{x,y}$.

(i) Each of the $(k+t)$ elements of $h(x, y)$ are R_h-accessible from every other element, and hence we must have that $n \geq (k + t)$, since f is a p-morphism. Moreover, every element of $h(x, y)$ is R_v-accessible from every other element, with an additional t elements being R_v-reflexive. It follows that $m \geq (k + 2t)$, since f is a p-morphism.

(ii) Suppose that $t = 0$ and that $m > k = |h(x, y)|$. By the pigeon-hole principle, there is some $i, j < m$ and some $w \in h(x, y)$ such that $f(0, i) = f(0, j) = w$ and $i \neq j$. However, since f is a p-morphism, we have that $w R_v w$, contrary to our supposition that $t = 0$. Hence if $t = 0$ then we must have that $m \leq k$, as required.

(iii) Suppose that $k > 0$ and let $w_0 \in h(x, y)$ be such that $\neg w_0 R_v w_0$. Since $h(x, y)$ is a cluster, we have that $f(0, i) R_v^+ w_0$, for all $i < m$. Therefore, courtesy of f being a p-morphism, we may define a function $\rho : m \to n$ such that $f(\rho(i), i) = w_0$, for all $i < m$.

Moreover, $\rho : m \to n$ is injective, for otherwise there would be $i, j < m$ such that $i \neq j$ and $\rho(i) = \rho(j)$. Hence there would be $f(\rho(i), i) R_v f(\rho(j), j)$, which is to say that $w_0 R_v w_0$, contrary to its definition. Thus we must have that ρ is injective, and therefore $m \leq n$, as required.

($\Leftarrow$) Conversely, suppose that $n, m \leq \omega$ satisfy conditions (i)–(iii). Let $u_0, \dots, u_{k-1}$ be an enumeration of the R_v-irreflexive elements of $h(x, y)$ and let $v_0, \dots, v_{t-1}$ be an enumeration of the R_v-reflexive elements of $h(x, y)$.

We define a function $\sigma : m \to h(x, y)$, by taking,

$$\sigma(i) = \begin{cases} u_i & \text{if } i < k, \\ v_\ell & \text{if } i \geq k \text{ and } \ell = i - k \mod t. \end{cases}$$

This may be more intuitively understood by the following enumeration of values.

i	0	$\dots$	$k-1$	k	$\dots$	$k+t-1$	$k+t$	$\dots$	$k+2t-1$	$\dots$
$\sigma(i)$	u_0	$\dots$	u_{k-1}	v_0	$\dots$	v_{t-1}	v_0	$\dots$	v_{t-1}	$\dots$
	R_v-irreflexive			R_v-reflexive			R_v-reflexive			

We then have two cases to consider:

– If $k > 0$ and $m = \omega$, then by (iii), we must have that $n = \omega$. Furthermore, by (ii), we have that $t > 0$ and so there is some $v_0 \in h(x, y)$, reflexive in R_v. We first define a function $f^- : \mathbb{Z} \times \mathbb{Z} \to h(x, y)$, by taking

$$f^-(i, j) = \begin{cases} \sigma(j - i) & \text{if } j \geq i, \\ v_0 & \text{otherwise.} \end{cases}$$

Each u_i occurs exactly once in each column and exactly once in each row, while each v_i occurs infinitely often throughout. It is then straightforward to check that f^- is a p-morphism from $(\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq)$ onto $\mathfrak{F}_{x,y}$. By relabelling the domain, we may then define a p-morphism $f : \omega \times \omega \to h(x, y)$ by taking $f(i, j) = f^-(\eta(i), \eta(j))$, for all $i, j < \omega$, where $\eta : \omega \to \mathbb{Z}$ is any bijection from ω onto $\mathbb{Z}$.

– Otherwise, we may define a function $f : n \times m \to h(x, y)$, by taking

$$f(i, j) = \sigma(\ell), \qquad \text{where } \ell = i + j \mod m,$$

for all $i < n$ and $j < m$. Since $n \geq k + t$, each element of $h(x, y)$ occurs at least once in each row. Furthermore, since $m \geq k + 2t$, each u_i occurs exactly once in each column, while each v_i occurs at least twice in each column. It is then straightforward to check that f is a p-morphism from $(n, n^2) \times (m, \neq)$ onto $\mathfrak{F}_{x,y}$, as required.

In both cases, we have that $\mathfrak{F}_{x,y}$ is the p-morphic image of $(n, n^2) \times (m, \neq)$, which is to say that $(n, m) \in K(x, y)$, as required.

□

Thus, we see that the task of deciding whether an arbitrary finite frame is a frame for $\mathbf{S5} \times \mathbf{Diff}$ reduces to that of finding a solution in $(\lambda_h(x) : x \in X)$ and $(\lambda_v(y) : y \in Y)$, over $(\omega + 1)$, to a finite set of linear equations. Since each set of constraints obeys the property that if $(n, m) \in K(x, y)$ then $(\omega, m) \in K(x, y)$, we are free to choose $\lambda_v(y) = \omega$, for all $y \in Y$, thereby reducing the problem further.

Indeed, it remains only to check that there is no conflict between the upper bounds imposed by those clusters in which $t = 0$, and those horizontally adjoining them. This observation leads to the following theorem.

Theorem 5.23. *The finite frame problem for* $\mathbf{S5} \times \mathbf{Diff}$ *is decidable in polynomial time.*

Proof. Let $\mathfrak{F} = (W, R_h, R_v)$ be an arbitrary finite frame for $[\mathbf{S5}, \mathbf{Diff}]$; any frame that is not a frame for $[\mathbf{S5}, \mathbf{Diff}]$ can be checked and eliminated in polynomial time — indeed in time cubic in the size of $\mathfrak{F}$. By Proposition 5.17, $\mathfrak{F}$ can be described by a grid of clusters (X, Y, h).

Since both $\mathsf{Fr}\,\mathbf{S5}$ and $\mathsf{Fr}\,\mathbf{Diff}$ are closed under ultra-products and point-generated subframes, it follows from Theorem 2.5 that $\mathfrak{F}$ is a frame for $\mathbf{S5} \times \mathbf{Diff}$ if and only if $\mathfrak{F}$ is the p-morphic image of a product frame for $\mathbf{S5} \times \mathbf{Diff}$, and so it suffices to check whether $\mathfrak{F}$ is the p-morphic image of a product frame for $\mathbf{S5} \times \mathbf{Diff}$.

From the forgoing it is sufficient to check that there is no $x_0, x_1 \in X$ and $y \in Y$ such that $t_0 = 0$ and $k_0 < k_1 + 2t_1$, where

$$k_i = |\{w \in h(x_i, y) : \neg w R_v w\}| \qquad \text{and} \qquad t_i = |\{w \in h(x_i, y) : w R_v w\}|,$$

for $i = 0, 1$.

Since there are at most $|X| \times |X| \times |Y| \leq |W|^3$ possible comparisons to be made, each taking linear time, we may decide whether $\mathfrak{F}$ is a frame for $\mathbf{S5} \times \mathbf{Diff}$ in time bounded by some *cubic* function in the size of W. □

Hence, we find that, despite the non-finite axiomatisability of $\mathbf{S5} \times \mathbf{Diff}$, the rigid structure of the frames for $[\mathbf{S5}, \mathbf{Diff}]$ — described by a grid of clusters — allows for a polynomial time decision procedure to its the finite frame problem; a vast improvement over the NExpTime upper bound offered by Theorem 5.15.

Moreover, Theorem 5.21 hints at the types of axioms that would be required in order to axiomatise $\mathbf{S5} \times \mathbf{Diff}$, for it applies not only to finite frames, but countable frames in general. As in the proof of Theorem 5.23, an arbitrary countable frame for $[\mathbf{S5}, \mathbf{Diff}]$ is the p-morphic image of a product frame for $\mathbf{S5} \times \mathbf{Diff}$ if and only if its ultrafilter extension contains no two horizontally adjoining clusters whose constraints are incompatible, in the above sense.

If we could somehow axiomatise these countably many first-order constraints, then we would have a full description of $\mathbf{S5} \times \mathbf{Diff}$. Indeed, infinitely many such axioms are required, as suggested by Theorem 5.1.

5.5 The Finite Frame Problem for Diff × Diff

A similar approach may be used to characterise the finite frames for $\mathbf{Diff} \times \mathbf{Diff}$, taking $(n, m) \in K'(x, y)$ if and only if $\mathfrak{F}_{x,y}$ is the p-morphic image of the product frame $(n, \neq) \times (m, \neq)$. The following natural analogue of the Theorem 5.21 is proved similarly for $\mathbf{Diff} \times \mathbf{Diff}$.

Theorem 5.24. *Let $\mathfrak{F}$ be a countable frame for $[\mathbf{Diff}, \mathbf{Diff}]$ and let (X, Y, h) be the grid of clusters describing $\mathfrak{F}$. Then $\mathfrak{F}$ is the p-morphic image of a product frame for $\mathbf{Diff} \times \mathbf{Diff}$ if and only if there is some $\lambda_h : X \to (\omega + 1)$ and $\lambda_v : Y \to (\omega + 1)$, such that*

$$\big(\lambda_h(x), \lambda_v(y)\big) \in K'(x, y),$$

for all $x \in X$ and $y \in Y$.

Proof. Analogous to that of Theorem 5.21. □

However, the there is far more variation in the possible clusters that can arise in arbitrary frames for $[\mathbf{Diff}, \mathbf{Diff}]$ than there is variation in what can occur in arbitrary frames for $[\mathbf{S5}, \mathbf{Diff}]$. For example, while $K(x, y)$ is always non-empty for every $[\mathbf{S5}, \mathbf{Diff}]$-cluster, there are clusters for $[\mathbf{Diff}, \mathbf{Diff}]$ that need not be the p-morphic image of *any* product frame for **Diff** $\times$ **Diff**.

Let $\mathfrak{F} = (W, R_h, R_v)$ be a countable frame for $[\mathbf{Diff}, \mathbf{Diff}]$ and let (X, Y, h) be the grid of clusters describing $\mathfrak{F}$. Suppose that $x \in X$ and $y \in Y$, and let

$$k = |\{w \in h(x,y) : \neg wR_hw \wedge \neg wR_vw\}|, \qquad t = |\{w \in h(x,y) : wR_hw \wedge wR_vw\}|,$$
$$t_h = |\{w \in h(x,y) : wR_hw \wedge \neg wR_vw\}|, \quad \text{and} \quad t_v = |\{w \in h(x,y) : \neg wR_hw \wedge wR_vw\}|.$$

Lemma 5.25. *Suppose that $x \in X$ and $y \in Y$, and let $t_h, t_v \leq \omega$ be as defined above. If $(n, m) \in K'(x, y)$, then the following conditions hold:*

(i) *If $t_h > 0$ then $2m \leq n$,*

(ii) *If $t_v > 0$ then $2n \leq m$.*

Proof. Suppose that $(n, m) \in K'(x, y)$ and let $f : n \times m \to h(x, y)$ be a p-morphism from $(n, \neq) \times (m, \neq)$ onto $\mathfrak{F}_{x,y}$.

(i) Suppose that $t_h > 0$ and let $w_0 \in h(x, y)$ be such that $w_0 R_h w_0$ and $\neg w_0 R_v w_0$. Since $\mathfrak{F}_{x,y}$ is a cluster, we have that $f(0, i)R_v^+ w_0$, for all $i < m$. Therefore, courtesy of f being a p-morphism, we may define a function $\rho_0 : m \to n$ such that $f(\rho_0(i), i) = w_0$, for all $i < m$. Moreover, since $w_0 R_h w_0$ we may define a second function $\rho_1 : m \to n$ such that $\rho_1(i) \neq \rho_0(i)$ and $f(\rho_0(i), i) = w_0$, as well, for $i < m$.

Combining these two functions we obtain $\rho : 2 \times m \to n$, by taking $\rho(i, j) = \rho_i(j)$, for all $i < 2$ and $j < m$.

Moreover, $\rho : 2 \times m \to n$ is injective, for otherwise there would be $i, i' < 2$ and $j, j' < m$ such that $(i, j) \neq (i', j')$ and $\rho(i, j) = \rho(i', j')$. If $j \neq j'$ then there would be $f(\rho(i, j), j)R_v f(\rho(i', j'), j')$, which is to say that wR_vw, contrary to its definition. Otherwise $j = j'$ and so $i \neq i'$. Whence we deduce that $\rho_i(j) \neq \rho_{i'}(j)$, from our choice of ρ_1. Thus we must have that ρ is injective, and therefore $2m \leq n$, as required.

(ii) Suppose that $t_v > 0$ and let $w_0 \in h(x, y)$ be such that $w_0 R_v w_0$ and $\neg w_0 R_h w_0$. We may define an injective function $\rho : 2 \times n \to m$, such that $f(i, \rho(0, i)) = f(i, \rho(1, i)) = w_0$ for all $i < n$, analogous to that described above. It follows that $2n \leq m$, as required.

□

The two above conditions ensure that if $t_h, t_v > 0$ are both non-zero, then $K(x, y) \subseteq \{(\omega, \omega)\}$. That is to say that if $\mathfrak{F}_{x,y}$ contains both both an R_h-irreflexive element and an R_v-irreflexive element, then $\mathfrak{F}_{x,y}$ cannot be the p-morphic image of *any* finite product frame.

Indeed, consider the following-three-element cluster $\mathfrak{H} = (\{a, b, c\}, R_h, R_v)$, given by

$$\begin{aligned} R_h &= \{(x, y) : x, y \in \{a, b, c\} \text{ and } x \neq y\} \cup \{(a, a), (b, b)\}, \\ R_v &= \{(x, y) : x, y \in \{a, b, c\} \text{ and } x \neq y\} \cup \{(a, a), (c, c)\}. \end{aligned}$$

From Lemma 5.25 it follows that if $\mathfrak{H}$ is the p-morphic image of $(n, \neq) \times (m, \neq)$, then $2n \leq m$ and $2m \leq n$, which is to say that $n = m = \omega$. Hence $\mathfrak{H}$ cannot be the p-morphic image of any *finite* product frame for $\mathbf{Diff} \times \mathbf{Diff}$. However, this does not preclude the possibility that $\mathfrak{H}$ may be the p-morphic image of an *infinite* product frame for $\mathbf{Diff} \times \mathbf{Diff}$, as demonstrated by the following proposition.

Proposition 5.26. *$\mathfrak{H}$ is a frame for* $\mathbf{Diff} \times \mathbf{Diff}$.

Proof. We construct a p-morphism from $\mathfrak{F}_h \times \mathfrak{F}_v$ onto $\mathfrak{H}$, where $\mathfrak{F}_i = (\mathbb{Z}, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, for $i = h, v$. We first define two function $\zeta_1, \zeta_2 : \mathbb{Z} \times \mathbb{Z} \to \{a, b, c\}$, by taking

$$\zeta_1(i, j) = \begin{cases} b & \text{if } i = j, j+1, \\ a & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \zeta_2(i, j) = \begin{cases} c & \text{if } j = i, i+1, \\ a & \text{otherwise.} \end{cases}$$

Note that, with ζ_1, b occurs exactly twice in each row and exactly once in each column, with a occurring infinitely often throughout. With ζ_2, c occurs exactly once in each row and exactly twice in each column, with a, again, occurring infinitely throughout.
We then 'splice' together *two* copies of each of these functions, taking $f : \mathbb{Z} \times \mathbb{Z} \to \{a, b, c\}$ to be the function given by

$$f(n, m) = \begin{cases} \zeta_1(\frac{1}{2}n, \frac{1}{2}m) & \text{if } n \text{ and } m \text{ are both even,} \\ \zeta_2(\frac{1}{2}n, \frac{1}{2}(m+1)) & \text{if } n \text{ is even and } m \text{ is odd,} \\ \zeta_2(\frac{1}{2}(n+1), \frac{1}{2}m) & \text{if } n \text{ is odd and } m \text{ is even,} \\ \zeta_1(\frac{1}{2}(n+1), \frac{1}{2}(m+1)) & \text{if } n \text{ and } m \text{ are both odd.} \end{cases}$$

Note here that b occurs twice in each row and once in each column, while c occurs twice in each column and once in each row, with a filling the remaining space. It follows that f

is a p-morphism of $\mathfrak{F}_h \times \mathfrak{F}_v$ onto $\mathfrak{H}$. Furthermore, since $\mathfrak{F}_h \times \mathfrak{F}_v$ is a frame for **Diff** $\times$ **Diff**, it follows from Proposition 2.4 that $\mathfrak{H}$ is also a frame for **Diff** $\times$ **Diff**, as required. $\square$

It follows that $\mathfrak{H}$ is an example of a finite frame for **Diff** $\times$ **Diff** that is not the p-morphic image of *any* finite product frame for **Diff** $\times$ **Diff**. This behaviour stands quite apart from that of **S5** $\times$ **Diff**, in which every finite frame for **S5** $\times$ **Diff** is the p-morphic image of some finite product frame.

More peculiar still, is the case that arises in the absence of any element reflexive in both R_h and R_v.

Proposition 5.27. *Let $\mathfrak{F} = (W, R_h, R_v)$ be a* finite *frame for* $[\mathbf{Diff}, \mathbf{Diff}]$ *and let (X, Y, h) be the grid of clusters describing $\mathfrak{F}$. Suppose that $x \in X$ and $y \in Y$, and let $t_h, t_v, t < \omega$ be defined as above. If $t = 0$ and $t_h, t_v > 0$ then $K'(x, y) = \emptyset$.*

Proof. Suppose, to the contrary, that $K'(x, y) \neq \emptyset$, and that $(n, m) \in K'(x, y)$, for some $n, m \leq \omega$. By Lemma 5.25, we require that $2m \leq n$ and $2n \leq m$, which is to say that $n = m = \omega$, since $n, m > 0$. So let $f : \omega \times \omega \to h(x, y)$ be a p-morphism from $(\omega, \neq) \times (\omega, \neq)$ onto $\mathfrak{F}_{x,y}$.

We define a function $\Omega : \omega \to 2^\omega$ such that $j \in \Omega(i)$ if and only if $f(i, j)$ is R_h-reflexive. That is to say that

$$\Omega(i) = \big\{ j < \omega : f(i, j) R_h f(i, j) \text{ and } \neg f(i, j) R_v f(i, j) \big\},$$

for all $i < \omega$.

We note that $|\Omega(i)| = t_h < \omega$ is finite, for all $i < \omega$. For we may define a function $g_i : \Omega(i) \to \{w \in h(x, y) : w R_h w \wedge \neg x R_v w\}$ by taking $g_i(j) = f(i, j)$, for all $j \in \Omega(i)$.

It is straightforward to check that g_i is injective, for suppose to the contrary that there is some $j, j' \in \Omega(i)$ such that $j \neq j'$ and $g_i(j) = g_i(j')$. By definition we have that $f(i, j) = f(i, j')$. However, since f is a p-morphism we must have that $f(i, j) R_v f(i, j)$ contrary to the definition of $\Omega(i)$.

Furthermore, we have that g_i is surjective, for suppose that $w \in h(x, y)$ is such that $w R_h w$ and $\neg w R_v w$. Either $f(i, 0) = w$ or $f(i, 0) R_v w$ since $h(x, y)$ is a cluster. However, since f is a p-morphism there is some $j < \omega$ such that $0 \neq j$ and $f(i, j) = w$. Hence in either case there is some $j < \omega$ such that $g_i(j) = w$. Moreover, it follows from our choice of w that $j \in \Omega(i)$.

Let $S = \bigcup_{i=0}^{\ell} \Omega(i)$ be the union of the first $(\ell+1)$ instances of Ω, where $\ell = |h(x,y)|$, and let $s = \sup S$ be the supremum of S, which is guaranteed to be finite since S is a finite union of finite sets. Consult Figure 5.4, below, for intuitions.

$$
\begin{array}{rr|ccccccccc:ccl}
 & & \multicolumn{9}{c}{\overbrace{\hphantom{0\quad 1\quad 2\quad 3\quad 4\quad 5\quad 6\quad 7\quad 8}}^{(\ell+1)}} & & & \\
 & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
\hline
\multirow{7}{*}{$S\left\{\rule{0pt}{6em}\right.$} & 0 & \# & & & \# & & \# & & \# & & & \# & \\
 & 1 & & \# & & & \# & \# & & & & \# & & \\
 & 2 & & & \# & & & & \# & \# & \# & & & \\
 & 3 & \# & & \# & \# & \# & & \# & & & \# & & \\
 & 4 & & & \# & & \# & & & \# & \# & & & \\
 & 5 & & \# & & & & \# & & & & & \# & \\
 & 6 & \# & \# & & \# & & & \# & & \# & & & \\
\hdashline
s+1= & 7 & & & & & & & & & & & \# & \\
 & \vdots & & & & & & & & & & \# & & \ddots
\end{array}
$$

Figure 5.4: Illustration of $\omega \times \omega$ grid depicting distribution of R_h-reflexive points (#).

Now consider the row $(s+1)$, and note that, by definition, no R_h-reflexive points may occur within the first $(\ell+1)$ columns. However by the pigeon-hole principle there must be some $j, j' < \ell+1$ such that $j \neq j'$ and $f(s+1, j) = f(s+1, j')$. Since f is a p-morphism we must have that $f(s+1,j) R_h f(s+1,j')$, which is to say that $f(s+1,j)$ is R_h-reflexive, contrary to definition of s.

Hence we conclude that $\mathfrak{F}_{x,y}$ is not the p-morphic image of *any* product frame for **Diff** $\times$ **Diff** — finite or otherwise — and that $K'(x,y) = \emptyset$, as required. $\square$

Thus, we see that there are many finite clusters that are not the p-morphic image *any* product frame for **Diff** $\times$ **Diff**; namely those having at least one R_h-reflexive point and at least one R_v-reflexive point, with no points reflexive in both R_h and R_v.

A full treatment of each of the fifteen possible cluster types can be performed using the techniques described above. We shall not labour this point here, but instead include for completeness the linear constraints imposed by each of the possible cases, given below in Table 5.2.

k	t_h	t_v	t	$K(x, y)$
0	0	0	>0	$\{(n, m) : n \geq 2t \text{ and } m \geq 2t\}$
0	0	>0	0	$\{(n, m) : n = t_v \text{ and } m \geq 2t_v\}$
0	0	>0	>0	$\{(n, m) : n \geq t_v + 2t \text{ and } m \geq 2t_v + 2t \text{ and } 2n \leq m\}$
0	>0	0	0	$\{(n, m) : n \geq 2t_h \text{ and } m = t_h\}$
0	>0	0	>0	$\{(n, m) : n \geq k + 2t_h + 2t \text{ and } m \geq k + t_h + 2t \text{ and } 2m \leq n\}$
0	>0	>0	0	$\emptyset$
0	>0	>0	>0	$\{(\omega, \omega)\}$
>0	0	0	0	$\{(k, k)\}$
>0	0	0	>0	$\{(n, n) : n \geq k + 2t\}$
>0	0	>0	0	$\emptyset$
>0	0	>0	>0	$\{(\omega, \omega)\}$
>0	>0	0	0	$\emptyset$
>0	>0	0	>0	$\{(\omega, \omega)\}$
>0	>0	>0	0	$\emptyset$
>0	>0	>0	>0	$\{(\omega, \omega)\}$

Table 5.2: Table of linear constraints associated with **Diff** $\times$ **Diff**.

5.6 Discussion

In Section 6.1, we showed that no logic between $\mathbf{K}\times\mathbf{Diff}$ and $\mathbf{S5}\times\mathbf{Diff}$ can be axiomatised using only finitely many variables. In particular we note that $\mathbf{Diff}\times\mathbf{Diff}$ is not finitely axiomatisable. In a related study, conducted by Kudinov et al. [70]†, the authors consider the logic of *Hamming spaces* — that is, frames of the form (A^n, H), where A is some finite alphabet, and uHv if and only if u and v differ only by a single letter, for all finite words $u, v \in A^n$ of length $n < \omega$.

They show that, for a given infinite alphabet A, and $n > 0$, the logic $\mathsf{Log}(A^n, H)$ is not axiomatisable using only finitely many variables. However, in the case where $n = 1$, this is equivalent to the claim that $\mathsf{Log}(A, \neq)$ is not axiomatisable using only finitely many variables. As a corollary, the authors deduce that $\mathsf{Log}((A, \neq) \times (B, \neq))$ is not finitely axiomatisable, whenever A is infinite and B is non-empty.

While these results are of a similar character to those described here, they differ in that they relate only to those logics characterised by a single product frame, whereas we are concerned chiefly with logics characterised by a wider class of product frames, in which the first component need not be a difference frame. Indeed, it is easily seen that each of the logics $\mathsf{Log}((A, \neq) \times (B, \neq))$ lie outside the remit of our Theorem 5.1, and that their results do not extend to those covered here.

Section 5.3, provides a modest extension of Theorem 3.10, by exposing a large class of logics that are product matching, without necessarily fulfilling the requirement of being Horn-axiomatisable.

While being fairly limited in its scope — applicable, as it is, only to products whose first component is **Alt** — it is suspected that this approach can be generalised to cover a range of logics that impose some finite bound on the number of possible successors each world may have. Let $\mathbf{Alt}(t)$ denote the logic characterised by all those unimodal frames in which every world has no more than $t < \omega$ distinct R-successors.

Question 5.28. Are $\mathbf{Alt}(t)$ and L product matching, whenever L is canonical?

While the decidability of the finite frame problem for logics that are not finitely axiomatisable is interesting in its own right, a full understanding of the structure of their

†Reproduced from an earlier publication [69], available only in Russian.

finite frames provides a preliminary stage in the investigation of possible axiom schema, which may serve to fully axiomatise the logic. This is the current state of affairs that persists with $\mathbf{K4.3} \times \mathbf{S5}$, which despite having a full description of its finite frames, resists all current attempts at an explicit axiomatisation [76].

Chapter 6

Finite Model Properties

As discussed above in Section 2.1.2, every finitely axiomatisable logic having the finite model property is decidable. However, neither the finite model property nor a finite axiomatisation alone is sufficient to guarantee decidability, nor are they both necessary. The study of finite model properties is therefore appealing in its own right. Despite the common coNExpTime-completeness of each of the logics $\mathbf{S5} \times \mathbf{S5}$, $\mathbf{S5} \times \mathbf{Diff}$ and $\mathbf{Diff} \times \mathbf{Diff}$, discussed Section 4.3, the three differ greatly with respect to their finite model properties. In this chapter we consider the finite model properties for each of these product logics.

It is well-established that $\mathbf{S5} \times \mathbf{S5}$ enjoys exponential product model property, and indeed even the exponential *square* model property. In Section 6.1 we describe an interval of bimodal logics extending $[\mathbf{wK5}, \mathbf{wK5}]$ that cannot be characterised by *any* class of finite frames. Included in this interval are the logics $\mathbf{Diff} \times \mathbf{Diff}$ and $\mathbf{Diff} \times^{sq} \mathbf{Diff}$ whose relationship to the two-variable fragment of first-order logic with counting quantifiers $\mathcal{C}^2$ has already been discussed in Chapter 4. While the lack of any finite model property for $\mathcal{C}^2$ is well-established, the results presented here do not follow from the first-order case.

More surprising is the case of $\mathbf{S5} \times \mathbf{Diff}$. Since the two-variable, substitution-free fragment of $\mathcal{C}^2$ having only quantifiers from among $\{\exists_{>0}x, \exists_{>0}y, \exists_{>1}y\}$ lacks the finite model property — as evidenced by the formula given in (4.1) — it follows that $\mathbf{S5} \times \mathbf{Diff}$ cannot be characterised by any class of finite *square* frames.

However, it is show in Section 6.2 that, not only does $\mathbf{S5} \times \mathbf{Diff}$ enjoy the finite model property, but it also enjoys the stronger exponential *product* model property.

6.1 Products Lacking the Abstract fmp

It is well-known that, while the two variable fragment of first-order logic enjoys the exponential finite model property, the addition of counting quantifiers introduces satisfiable formulas that cannot be satisfied in any finite model [52]. An example of such a formula is given above in (4.1).

Owing to the connections that the fragment $\mathcal{C}^2$ shares with the logics $\mathbf{Diff} \times \mathbf{Diff}$ and $\mathbf{S5} \times \mathbf{Diff}$, discussed in Section 4.3, we may extend this result to products, by way of the following theorem.

Theorem 6.1. *Let L be any modal logic such that $(\omega, \omega^2) \in \mathsf{Fr}\, L$. Then $L \times \mathbf{Diff}$ does not enjoy the* square *product fmp.*

Proof. Consider the following modal equivalent to the first-order formula given in (4.1):

$$\theta_\infty \;:=\; \Box_v^+ \Diamond_h p \wedge \Box_h^+ \Diamond_v^{\leq 1} p \wedge \Diamond_h^+ \Box_v^+ \neg p. \tag{6.1}$$

Suppose, to the contrary, that θ_∞ is satisfiable in some finite square product model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathsf{Fr}\, L$ and $\mathfrak{F}_v = (W_v, \neq) \in \mathsf{Fr}\, \mathbf{Diff}$ are such that $|W_h| = |W_v| < \omega$. We define a function $g : W_v \to W_h$ by choosing $g(y) \in \{x \in W_h : (x, y) \in \mathfrak{V}(p)\}$, for all $y \in W_v$. It follows from (6.1) that g is injective but not surjective, which is impossible since we assumed both $\mathfrak{F}_h$ and $\mathfrak{F}_v$ to be finite.

Furthermore, we note that θ_∞ is $L \times \mathbf{Diff}$-satisfiable, as evidenced by the model $\mathfrak{M} = (\mathfrak{G}_h \times \mathfrak{G}_v, \mathfrak{V})$, where

$$\mathfrak{G}_h = (\omega, \omega^2) \in \mathsf{Fr}\, L, \qquad \mathfrak{G}_v = (\omega, \neq) \in \mathsf{Fr}\, \mathbf{Diff}, \qquad \mathfrak{V}(p) = \{(n+1), n) : n < \omega\}.$$

It is straightforward to check that $\mathfrak{M}, (0,0) \models \theta_\infty$.

Hence θ_∞ is an example of an $L \times \mathbf{Diff}$-satisfiable formula that cannot be satisfied in any finite square product frame for $L \times \mathbf{Diff}$. That is to say that $L \times \mathbf{Diff}$ does not possess the finite square product model property, as required. $\Box$

In particular, it follows that neither $\mathbf{Diff} \times \mathbf{Diff}$ nor $\mathbf{S5} \times \mathbf{Diff}$ possess the finite square product model property. Furthermore, by taking the conjunction of θ_∞, given in (6.1), with the following formula,

$$\Box_h^+ \Diamond_v^{=1} q \wedge \Box_v^+ \Diamond_h^{=1} q, \tag{6.2}$$

we are able to show that **Diff** $\times$ **Diff** lacks even the *product fmp*. For it is similarly straightforward to show that if (6.2) is satisfied in a product frame for **Diff** $\times$ **Diff** then such a frame should be square. Thus, if the conjunction of (6.1) and (6.2) is to be satisfied in a product frame for **Diff** $\times$ **Diff**, then such a frame should be square, and by virtue of Theorem 6.1, must be infinite.

Corollary 6.2. **Diff** $\times$ **Diff** *does not possess the product fmp.*

Were it the case that every finite frame for **Diff** $\times$ **Diff** is a p-morphic image of a finite *product* frame then we would also conclude that **Diff** $\times$ **Diff** did not possess the *abstract* fmp; as every formula satisfiable in a finite frame for **Diff** $\times$ **Diff** could be satisfied in a finite product frame.

However, as described above in Section 5.4, there are finite frames for **Diff** $\times$ **Diff** that are not the p-morphic image of *any* finite product frame. It therefore requires a stronger argument to demonstrate that **Diff** $\times$ **Diff** lacks the *abstract* fmp. Here, we prove a more general theorem concerning all normal extensions of $[\mathbf{wK5}, \mathbf{wK5}]$, having $(\omega, \neq) \times (\omega, \neq)$ among their frames.

Theorem 6.3. *Let L be any bimodal logic such that:*

- $[\mathbf{wK5}, \mathbf{wK5}] \subseteq L$,
- $(\omega, \neq) \times (\omega, \neq)$ *is a frame for L.*

Then L does not possess the abstract *finite model property.*

Let φ_∞ be the conjunction of the following formulas:

$$\Diamond_h \Diamond_v (c \wedge \neg d \wedge \Box_h \neg c \wedge \Box_v \neg d), \tag{6.3}$$

$$\Box_h \Diamond_v (c \wedge \neg d \wedge \Box_h \neg c), \tag{6.4}$$

$$\Box_v \Diamond_h (d \wedge \neg c \wedge \Box_v \neg d). \tag{6.5}$$

Lemma 6.4. *Let $\mathfrak{F} = (W, R_h, R_v)$ be any frame for* $[\mathbf{wK5}, \mathbf{wK5}]$. *If φ_∞ is satisfiable in $\mathfrak{F}$ then $\mathfrak{F}$ must be infinite.*

Proof. Let $\mathfrak{F} = (W, R_h, R_v)$ be as described and suppose that $\mathfrak{M}, r \models \varphi_\infty$, for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ based on $\mathfrak{F}$, with $r \in W$. We then define, inductively, four infinite sequences:

$$\langle x_k \in W : k < \omega \rangle, \qquad \langle y_k \in W : k < \omega \rangle, \qquad \langle u_k \in W : k < \omega \rangle, \quad \text{and} \quad \langle v_k \in W : k < \omega \rangle,$$

such that $\mathfrak{M}, u_0 \models \Box_v \neg d$, and for all $k < \omega$:

(gen1) $rR_h x_k$ and $rR_v y_k$,

(gen2) $x_k R_v u_k$ and $x_{k+1} R_v v_k$,

(gen3) $y_k R_h v_k$ and $y_k R_h u_k$,

(gen4) $\mathfrak{M}, u_k \models c \land \neg d \land \Box_h \neg c$,

(gen5) $\mathfrak{M}, v_k \models d \land \neg c \land \Box_v \neg d$.

At this stage we do not assume that all the points are distinct from one another.

Firstly, by the formula given in (6.3), there is some $x_0, u_0 \in W$ such that $rR_h x_0$, $x_0 R_v u_0$ and $\mathfrak{M}, u_0 \models c \land \neg d \land \Box_h \neg c \land \Box_v \neg d$. Then by ($com^r$), there is some $y_0 \in W$ such that $rR_v y_0$ and $y_0 R_h u_0$. Whence it follows from (6.5) that there is some $v_0 \in W$ such that $y_0 R_h v_0$ and $\mathfrak{M}, v_0 \models d \land \neg c \land \Box_v \neg d$.

Now suppose we have already defined x_k, y_k, u_k, v_k, for some $k < \omega$. By **(gen1)** and **(gen3)**, we have that $rR_v y_k$ and $y_k R_h v_k$. Then by (com^l), there is some $x_{k+1} \in W$ such that $rR_h x_{k+1}$ and $x_{k+1} R_v v_k$. Whence it follows from (6.4) that there is some $u_{k+1} \in W$ such that $x_{k+1} R_v u_{k+1}$ and $\mathfrak{M}, u_{k+1} \models c \land \neg d \land \Box_h \neg c$.

Now by (com^r), there is some $y_{k+1} \in W$ such that $rR_v y_{k+1}$ and $y_{k+1} R_h u_{k+1}$. Whence it follows from the formula given in (6.5) that there is some $v_{k+1} \in W$ such that $y_{k+1} R_h v_{k+1}$ and $\mathfrak{M}, v_{k+1} \models d \land \neg c \land \Box_v \neg d$.

Hence, by induction on the length, we may extend each of the four sequences indefinitely, as depicted in Figure 6.1.

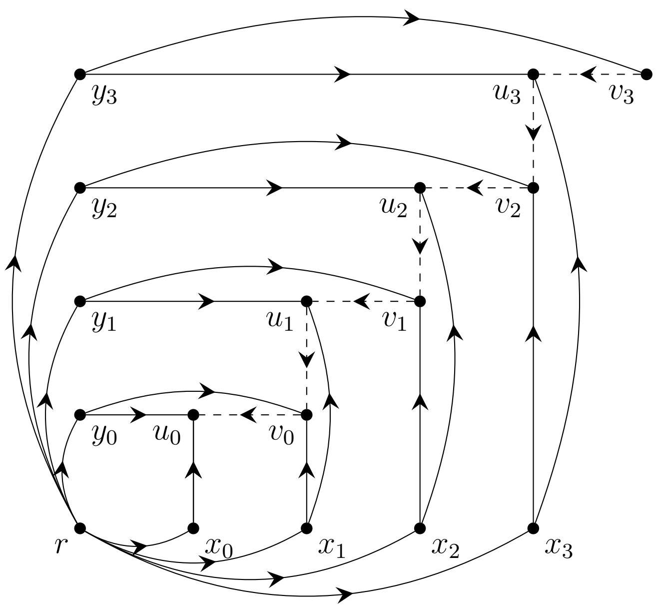


Figure 6.1: Illustration of the model generated by φ_∞.

We now show that each of the u_k are distinct. To this end, we define inductively a sequence of formulas by taking $\psi_0 := \Box_v \neg d$, and for $k < \omega$:

$$\psi_{k+1} \ := \ \Diamond_v\big(d \wedge \Diamond_h(c \wedge \psi_k)\big).$$

We claim that, for all $k < \omega$:

$$\mathfrak{M}, u_k \models \psi_k \wedge \neg\psi_{k+1}. \tag{6.6}$$

Indeed, it is immediate from the definitions that $\mathfrak{M}, u_0 \models \psi_0 \wedge \neg\psi_1$. So suppose that $\mathfrak{M}, u_k \models \psi_k \wedge \neg\psi_{k+1}$, for some $k < \omega$. By **(gen4)**–**(gen5)**, we must have that $u_i \neq v_j$, for all $i, j < \omega$. Therefore it follows from **(gen2)**–**(gen3)** and the weak-Euclideanness of both R_h and R_v that $v_k R_h u_k$ and $u_{k+1} R_v v_k$. Whence by **(gen4)**–**(gen5)**, we deduce that $\mathfrak{M}, u_{k+1} \models \psi_{k+1}$.

Now suppose that $\mathfrak{M}, v_{k+1} \models \psi_{k+2}$. Then there is some $u, v \in W$ such that $u_{k+1} R_v v$, $v R_h u$, $\mathfrak{M}, v \models d$, and $\mathfrak{M}, u \models c \wedge \psi_{k+1}$. It then follows from **(gen2)**–**(gen5)** and the weak-Euclideanness of both R_h and R_v that $v = v_k$ and $u = u_k$. This is to say that $\mathfrak{M}, u_k \models \psi_{k+1}$, contrary to our inductive hypothesis. Hence, $\mathfrak{M}, u_{k+1} \models \neg\psi_{k+2}$, as required.

It then follows from (6.6) that each of the $u_k \in W$ must be distinct. Being such, we must have that $\mathfrak{F}$ is infinite, as required. $\square$

Furthermore, we note that φ_∞ is L-satisfiable, as evidenced by the model $\mathfrak{M} = (\mathfrak{H}, \mathfrak{V})$, where $\mathfrak{H} = (\omega, \neq) \times (\omega, \neq) \in \mathsf{Fr}\, L$ and

$$\mathfrak{V}(c) = \{(n, n) : 0 < n < \omega\},$$
$$\mathfrak{V}(d) = \{(n + 1, n) : 0 < n < \omega\}.$$

It is straightforward to check that $\mathfrak{M}, (0, 0) \models \varphi_\infty$.

Hence it follows that φ_∞ is an example of an L-satisfiable formula that cannot be satisfied in any finite frame for L, abstract or otherwise. This is to say that L does not possess the abstract finite model property, thereby completing the proof of Theorem 6.3.

As an immediate consequence of Theorem 6.3, we have that neither $[\mathbf{Diff}, \mathbf{Diff}]$ nor $\mathbf{Diff} \times \mathbf{Diff}$ posses the abstract fmp.

Corollary 6.5. *Neither* $[\mathbf{Diff}, \mathbf{Diff}]$ *nor* $\mathbf{Diff} \times \mathbf{Diff}$ *possess even the abstract fmp.*

6.2 Quasimodels

In Theorem 6.1, we noted that neither $\mathbf{Diff} \times \mathbf{Diff}$ nor $\mathbf{S5} \times \mathbf{Diff}$ enjoy the *square* product fmp. However, while $\mathbf{Diff} \times \mathbf{Diff}$ lacks both the *product* fmp and the *abstract* fmp, our results, thus far, have yet to say anything about whether $\mathbf{S5} \times \mathbf{Diff}$ enjoys the *product* fmp.

The current literature is rife with examples of products logics possessing the *abstract* fmp while lacking the rigidity of the stronger *product* fmp [39, 38]. However it is less clear that this notion of the *square* product fmp — so intrinsic to first-order logics — is any more of a restriction than the more well-examined product fmp.

In this section we employ a version of the method of *quasimodels* [134, 38] to show that not only is $\mathbf{S5} \times \mathbf{Diff}$ characterised by its finite models, but that every $\mathbf{S5} \times \mathbf{Diff}$-satisfiable formula can be satisfied in a *product* model whose size is bounded by a *singly* exponential function in the length of the formula.

Theorem 6.6. $\mathbf{S5} \times \mathbf{Diff}$ *has the exponential product fmp.*

We follow a similar approach to that described in [38, Theorem 5.22], which provides an alternative demonstration that $\mathbf{S5} \times \mathbf{S5}$ enjoys the exponential product fmp. However the added expressivity of $\mathbf{S5} \times \mathbf{Diff}$ — in particular, its ability to express certain counting modalities — necessitates a more elaborate strategy.

Let $\varphi \in \mathcal{ML}_2$ be an arbitrary bimodal formula of size $n = |\mathrm{sub}(\varphi)|$, and define a *type* for φ to be any subset $t \subseteq \mathrm{sub}(\varphi)$ that is *Boolean-saturated* in the sense that:

- $\neg\psi \in t$ if and only if $\psi \notin t$, for all $\neg\psi \in \mathrm{sub}(\varphi)$;
- $\psi_1 \wedge \psi_2 \in t$ if and only if $\psi_1 \in t$ and $\psi_2 \in t$, for all $\psi_1 \wedge \psi_2 \in \mathrm{sub}(\varphi)$.

Definition 6.7. We define a **Diff**-*quasistate* (or simply, a *quasistate*) for φ to be a pair (T, S) such that:

(qs1) T is a non-empty set of *types* for φ, and S is a binary relation on T,

(qs2) For all $t, t' \in T$, we have that tS^+t',

(qs3) *($\Diamond_v$-saturation)* For all $t \in T$ and $\Diamond_v\alpha \in \mathrm{sub}(\varphi)$,

$$\Diamond_v\alpha \in t \qquad \Longleftrightarrow \qquad \exists t' \in T;\ tSt' \text{ and } \alpha \in t'.$$

It is worth noting here that, since there can be at most 2^n distinct types for φ, the size of each quasistate is bounded by a singly exponential function of the size of φ.

A *basic structure* for φ will be a pair $(W, \boldsymbol{q})$ where W is a non-empty set and $\boldsymbol{q}$ is a function associating each $w \in W$ with a quasistate $\boldsymbol{q}(w) = (T_w, S_w)$. An *(indexed) run through* $(W, \boldsymbol{q})$ will be any pair (r, i), where r is a function associating each $w \in W$ with a type $r(w) \in T_w$, and i is an index, used to distinguish otherwise identical runs through $(W, \boldsymbol{q})$. That is to say that a set of indexed runs may contain arbitrarily many pairs (r, i) and (r', i'), where $r(w) = r'(w)$ for all $w \in W$. This is a minor technical point, and we shall, hereafter, associate each indexed run with the function described by its first argument, forgoing reference to the index.

Definition 6.8. An $\mathbf{S5} \times \mathbf{Diff}$-*quasimodel for* φ (or simply a *quasimodel* for φ) is a tuple $\mathfrak{Q} = (W, \boldsymbol{q}, \mathfrak{R})$ such that:

(qm1) $(W, \boldsymbol{q})$ is a basic structure for φ, and $\mathfrak{R}$ is an set of indexed runs through $(W, \boldsymbol{q})$,

(qm2) There is some $w_0 \in W$ and $r_0 \in \mathfrak{R}$ such that $\varphi \in r_0(w_0)$,

(qm3) *(coherence)* For all $r \in \mathfrak{R}$, $w \in W$ and $\Diamond_h \alpha \in \mathrm{sub}(\varphi)$

$$\exists v \in W;\ \alpha \in r(v) \qquad \Longrightarrow \qquad \Diamond_h \alpha \in r(w),$$

(qm4) *(saturation)* For all $r \in \mathfrak{R}$, $w \in W$ and $\Diamond_h \alpha \in \mathrm{sub}(\varphi)$

$$\Diamond_h \alpha \in r(w) \qquad \Longrightarrow \qquad \exists v \in W;\ \alpha \in r(v),$$

(qm5) For all $r \in \mathfrak{R}$, $w \in W$ and $t \in T_w$, if $r(w) S_w t$ then there is some $r' \in \mathfrak{R}$ such that $r \neq r'$ and $r'(w) = t$,

(qm6) For all $w \in W$ and $r, r' \in \mathfrak{R}$, if $r \neq r'$ then $r(w) S_w r'(w)$.

The following lemmas show that, although our quasimodels are not $\mathbf{S5} \times \mathbf{Diff}$-models in their own right, they do retain enough information about a full model to capture the notion of $\mathbf{S5} \times \mathbf{Diff}$-satisfiability.

Lemma 6.9. *If φ is $\mathbf{S5} \times \mathbf{Diff}$-satisfiable then φ has a quasimodel.*

Proof. Suppose that φ is $\mathbf{S5} \times \mathbf{Diff}$-satisfiable. Then $\mathfrak{M}, (r_h, r_v) \models \varphi$ for some product model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, W_h \times W_h) \in \mathsf{Fr}\,\mathbf{S5}$ and $\mathfrak{F}_v = (W_v, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$.

With every pair $(x, y) \in W_h \times W_v$, we associate the type

$$\boldsymbol{t}(x, y) = \{\alpha \in \mathrm{sub}(\varphi) : \mathfrak{M}, (x, y) \models \alpha\},$$

and define a basic structure $(W_h, \boldsymbol{q})$, by taking $\boldsymbol{q}(x) = (T_x, S_x)$, for all $x \in W_h$, where

- $T_x = \{\boldsymbol{t}(x, y) : y \in W_v\}$,
- $t S_x t'$ if and only if there is some $y, y' \in W_v$ such that $\boldsymbol{t}(x, y) = t$, $\boldsymbol{t}(x, y') = t'$ and $y \neq y'$.

It is straightforward to check that $\boldsymbol{q}(x)$ is a quasistate, for each $x \in W_h$. Furthermore, for each $y \in W_v$, we define a function $r_y : W_h \to 2^{\mathrm{sub}(\varphi)}$, by taking

$$r_y(x) = \boldsymbol{t}(x, y),$$

for all $x \in W_h$. We then take $\mathfrak{R} = \{(r_y, y) : y \in W_v\}$.

It remains to check that $(W_h, \boldsymbol{q}, \mathfrak{R})$ is a quasimodel for φ.

- For **(qm2)**, we have that $\mathfrak{M}, (r_h, r_v) \models \varphi$ and so $\varphi \in \boldsymbol{t}(r_h, r_v) = r_{r_v}(r_h)$, as required.
- For **(qm3)** and **(qm4)**, suppose $r_y \in \mathfrak{R}$, $x \in W_h$ and $\Diamond_h \alpha \in \mathrm{sub}(\varphi)$ then

$$\begin{aligned} \Diamond_h \alpha \in r_y(x) &\iff \Diamond_h \alpha \in \boldsymbol{t}(x, y), \\ &\iff \mathfrak{M}, (x, y) \models \Diamond_h \alpha, \\ &\iff \exists x' \in W_h;\ \mathfrak{M}, (x', y) \models \alpha, \\ &\iff \exists x' \in W_h;\ \alpha \in \boldsymbol{t}(x', y), \\ &\iff \exists x' \in W_h;\ \alpha \in r_y(x'). \end{aligned}$$

- For **(qm5)**, suppose $r_y \in \mathfrak{R}$, $x \in W_h$ and $t \in T_x$ are such that $r_y(x) S_x t$. It follows that $t = \boldsymbol{t}(x, z)$ for some $z \in W_v$ such that $y \neq z$. Thus, we may choose $z \in \mathfrak{R}$ so that $r_y \neq r_z$ and $r_z(x) = t$, as required.
- For **(qm6)**, suppose that $x \in W_h$ and $r_y, r_z \in \mathfrak{R}$ are such that $r_y \neq r_z$. Hence $y \neq z$ and so it follows immediately that $r_y(x) = \boldsymbol{t}(x, y) S_x \boldsymbol{t}(x, z) = r_z(x)$, as required.

Hence, $(W_h, \boldsymbol{q}, \mathfrak{R})$ is a suitable quasimodel for φ, as required. $\square$

Conversely, we show that every quasimodel retains enough information for us to reconstruct a full product model satisfying φ, of size proportional to the size of the quasimodel.

Lemma 6.10. *If $\mathfrak{Q} = (W, \boldsymbol{q}, \mathfrak{R})$ is a quasimodel for φ is then φ is satisfiable in a* $\mathbf{S5} \times \mathbf{Diff}$ *model of size bounded by $|W| \cdot |\mathfrak{R}|$.*

Proof. Suppose that $\mathfrak{Q} = (W, \boldsymbol{q}, \mathfrak{R})$ is a quasimodel for φ. We define a new model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, by taking

$$\mathfrak{F}_h = (W, W^2) \in \mathsf{Fr}\,\mathbf{S5}, \qquad \mathfrak{F}_v = (\mathfrak{R}, \neq) \in \mathsf{Fr}\,\mathbf{Diff}, \quad \text{and} \quad \mathfrak{V}(p) = \{(w, r) : p \in r(w)\},$$

for all propositional variables $p \in \mathrm{sub}(\varphi)$. Clearly $\mathfrak{F}_h \times \mathfrak{F}_v$ is a frame for $\mathbf{S5} \times \mathbf{Diff}$ of the prescribed size, and so it remains to check that $\mathfrak{M}$ is a model for φ.

We show that for all $w \in W$, $r \in \mathfrak{R}$, and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}, (w, r) \models \psi \qquad \Longleftrightarrow \qquad \psi \in r(w). \tag{I.H.}$$

The cases where ψ is a propositional variable or a Boolean combination of smaller formulas are trivial and follow immediately from the definitions and the fact that types are Boolean saturated. So suppose that ψ is of the form $\Diamond_h \alpha$ for some $\alpha \in$ sub(φ). Then we have that,

$$\begin{aligned}
\mathfrak{M}, (w, r) \models \Diamond_h \alpha &\iff \exists w' \in W;\ \mathfrak{M}, (w', r) \models \alpha, \\
&\iff \exists w' \in W;\ \alpha \in r(w') \quad \text{by (I.H.)}, \\
&\iff \Diamond_h \alpha \in r(w) \quad \text{by } \textbf{(qm3)} \text{ and } \textbf{(qm4)}.
\end{aligned}$$

Now suppose that ψ of the form $\Diamond_v \alpha$, for some $\alpha \in \mathrm{sub}(\varphi)$. Then we have that,

$$\begin{aligned}
\mathfrak{M}, (w, r) \models \Diamond_v \alpha &\implies \exists r' \in \mathfrak{R};\ r \neq r' \text{ and } \mathfrak{M}, (w, r') \models \alpha, \\
&\implies \exists r' \in \mathfrak{R};\ r \neq r' \text{ and } \alpha \in r'(w) \quad \text{by (I.H.)}, \\
&\implies \exists r'(w) \in T_w;\ r'(w) S_w r'(w) \text{ and } \alpha \in r'(w) \quad \text{by } \textbf{(qm6)}, \\
&\implies \Diamond_v \alpha \in r(w) \quad \text{by } \textbf{(qs3)}.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\Diamond_v \alpha \in r(w) &\implies \exists t \in T_w;\ r(w) S_w t \text{ and } \alpha \in t \quad \text{by } \textbf{(qs3)}, \\
&\implies \exists r' \in \mathfrak{R};\ r \neq r' \text{ and } \alpha \in r'(w) \quad \text{by } \textbf{(qm5)}, \\
&\implies \exists r' \in \mathfrak{R};\ r \neq r' \text{ and } \mathfrak{M}, (w, r') \models \alpha \quad \text{by I.H.}, \\
&\implies \mathfrak{M}, (w, r) \models \Diamond_v \alpha.
\end{aligned}$$

Hence, (I.H.) holds for all subformulas of φ, while by **(qm4)**, there is some $w_0 \in W$ and some $r_0 \in \mathfrak{R}$ such that $\varphi \in r_0(w_0)$. In particular, we have that $\mathfrak{M}, (w_0, r_0) \models \varphi$, as required. $\square$

Hence, to prove that $\mathbf{S5} \times \mathbf{Diff}$ has the finite product model property, it is sufficient to show that every large quasimodel for φ can be effectively 'pruned' to yield a smaller quasimodel whose size is at most exponential in the size of φ.

Lemma 6.11. *If φ has a quasimodel, then φ has a quasimodel that is at most exponential in the size of φ.*

Proof. Suppose that $\mathfrak{Q} = (W, \boldsymbol{q}, \mathfrak{R})$ is a quasimodel for φ. It follows from **(qm2)** that there is some $w_0 \in W$ and $t_0 \in T_{w_0}$ such that $\varphi \in t_0$. By **(qm5)**, for each $t \in T_{w_0}$ we may fix some run $s_t \in \mathfrak{R}$ such that $s_t(w_0) = t$. Let $\mathfrak{S} = \{s_t : t \in T_{w_0}\}$ be the set comprising all such runs.

Now, courtesy of **(qm4)**, for all $t \in T_{w_0}$ and $\Diamond_h \alpha \in t$, we may fix some $v_{(t,\alpha)} \in W$ such that $\alpha \in s_t(v_{(t,\alpha)})$. We then define a new basic structure $(W_1, \boldsymbol{q}_1)$, by taking

$$\begin{aligned} W_1 &= \{w_0\} \cup \{v_{(t,\alpha)} \in W : t \in T_{w_0} \text{ and } \Diamond_h \alpha \in t\}, \\ \boldsymbol{q}_1(w) &= \boldsymbol{q}(w), \quad \text{for all } w \in W_1. \end{aligned}$$

Clearly, by design, each of the runs $s \in \mathfrak{S}$ are coherent and saturated with respect to $(W_1, \boldsymbol{q}_1)$. However, $\mathfrak{S}$ need not be plentiful enough to accommodate **(qm5)**; that is to say, there may be many types that are not witnessed by any run in $\mathfrak{S}$.

To remedy this, we extend $\mathfrak{S}$ to a 'small' subset $\mathfrak{R}_1$ of $\mathfrak{R}$ by choosing sufficiently many runs so as to satisfy **(qm5)**. Indeed, for each $w \in W_1$ and $t \in T_w$ let $r_{w,t} \in \mathfrak{R}$ be such that $r_{w,t}(w) = t$. Moreover, if tS_wt then let $r'_{w,t} \in \mathfrak{R}$ be such that $r_{w,t} \neq r'_{w,t}$ and $r_{w,t}(w) = t$. Take $\mathfrak{R}_1$ to be the set of all such runs.

Taking $\mathfrak{Q}_1 = (W_1, \boldsymbol{q}_1, \mathfrak{R}_1)$, it is straightforward to check that $\mathfrak{Q}_1$ satisfies all of the conditions **(qm1)**–**(qm6)**, except for the saturation condition **(qm4)**. Furthermore, $\mathfrak{Q}_1$ is finite, with

$$|W_1| \leq 1 + 2^n \cdot n \qquad \text{and} \qquad |\mathfrak{R}_1| \leq (1 + 2^n \cdot n) \cdot 2^{n+1}.$$

We now diverge from the techniques of [134, 38], by introducing a 'copy' of each quasistate $\boldsymbol{q}(u)$, for each pair of runs $(s, r) \in \mathfrak{S} \times \mathfrak{R}_1$, where $s(w_0) = r(w_0)$. The intuition being that since s and r coincide at $w_0 \in W_1$, we may coherently transpose the values of s and r at any $u \in W_1$. However, as to not risk unsaturating s, we provide multiple copies of each $u \in W_1$ to allow a single transposition in each quasistate.

We define a new basic structure $(W_2, \boldsymbol{q}_2)$, by taking

$$\begin{aligned} W_2 &= \{(u, s, r) \in W_1 \times \mathfrak{S} \times \mathfrak{R}_1 : s(w_0) = r(w_0)\}, \\ \boldsymbol{q}_2(u, r) &= \boldsymbol{q}_1(u) \quad \text{for all } (u, r) \in W_2. \end{aligned}$$

For each $r \in \mathfrak{R}_1$, we define a new run r' through $(W_2, \boldsymbol{q}_2)$ such that

$$r'(u, s_0, r_0) = \begin{cases} s_0(u) & \text{if } r \notin \mathfrak{S} \text{ and } r = r_0, \\ r_0(u) & \text{if } r \in \mathfrak{S} \text{ and } r = s_0, \\ r(u) & \text{otherwise,} \end{cases}$$

for all $u \in W_1$, $s_0 \in \mathfrak{S}$, and $r_0 \in \mathfrak{R}_1$. Let $\mathfrak{R}_2 = \{r' : r \in \mathfrak{R}_1\}$ be the collection of all such runs and define $\mathfrak{Q}_2 = (W_2, \boldsymbol{q}_2, \mathfrak{R}_2)$.

It is straightforward to check that

$$|W_2| \leq (1 + 2^n \cdot n)^2 \cdot 2^{2n+1} \qquad \text{and} \qquad |\mathfrak{R}_2| = |\mathfrak{R}_1| \leq (1 + 2^n \cdot n) \cdot 2^{n+1}, \tag{6.7}$$

and so it remains to show that $\mathfrak{Q}_2$ is a quasimodel for φ.

- Condition **(qm2)** follows immediately from the construction.
- For **(qm3)**, it is enough to note that $r'(u, s_0, r_0) = r(u)$ for all $(u, s_0, r_0) \in W_2$, unless r is either of s_0 or r_0, in which case the values are transposed. However, since s_0 and r_0 coincide at $w_0 \in W$, we can be sure that coherence is maintained.
- For **(qm4)**, let $r' \in \mathfrak{R}_2$, $(u, s_0, r_0) \in W_2$ and $\Diamond_h \alpha \in \mathrm{sub}(\varphi)$ be such that $\Diamond_h \alpha \in r'(u, s_0, r_0)$. This is to say that $\Diamond_h \alpha$ belongs to one of $r(u)$, $s_0(u)$ or $r_0(u)$. However, since each of these runs are both saturated and coherent over $(W, \boldsymbol{q})$, it follows that in all cases $\Diamond_h \alpha \in t$ where $t = r(w_0) \in T_{w_0}$. Hence there is some $v_{(t,\alpha)} \in W_1$ such that $\alpha \in s_t(v_{(t,\alpha)})$, for $s_t \in \mathfrak{S}$, which may or may not be s_0. Hence it follows that $r'(v_{(t,\alpha)}, s_t, r) = s_t(v_{(t,\alpha)})$.

  Thus there is some $(v_{(t,\alpha)}, s_t, r) \in W_2$ such that $\alpha \in r'(v_{(t,\alpha)}, s_t, r)$, as required.
- For both **(qm5)**–**(qm6)**, it is enough to note that the number of runs passing through each type of $\boldsymbol{q}_2(u, s_0, r_0)$ in $\mathfrak{Q}_2$ matches the number of runs chosen to pass through each

type of $\boldsymbol{q}_1(u)$ of $\mathfrak{Q}_1$. The only difference being that the values of s_0 and r_0 are transposed. Since $\mathfrak{Q}_1$ satisfies both **(qm5)** and **(qm6)**, so too must $\mathfrak{Q}_2$, as required.

Hence $\mathfrak{Q}_2$ is a quasimodel for φ, whose size is at most exponential in the size of φ, as required. $\square$

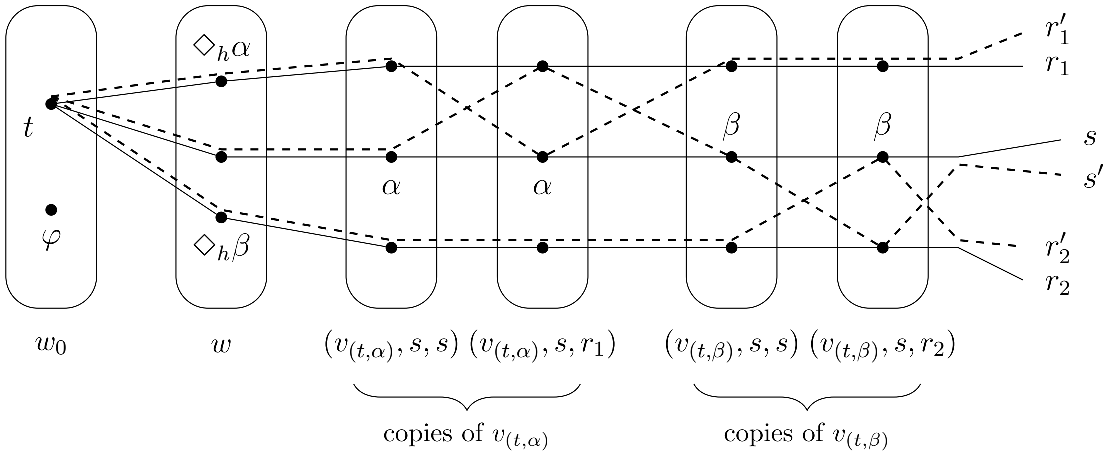


Figure 6.2: Illustration of transposing runs.

It then follows immediately from Lemmas 6.9–6.10 that every $\mathbf{S5} \times \mathbf{Diff}$-satisfiable formula φ can be satisfied in a product model for $\mathbf{S5} \times \mathbf{Diff}$, whose size is at most exponential in the size of φ. That is to say that $\mathbf{S5} \times \mathbf{Diff}$ has the exponential product fmp, thereby completing the proof of Theorem 6.6.

It is worth noting that in the process of finitizing our quasimodels, the size of W_2 is disproportionately larger than the size of $\mathfrak{R}_2$, as can be seen from (6.7). Indeed, since $\mathbf{S5} \times \mathbf{Diff}$ lacks the square product fmp, this is precisely as we would expect.

6.3 Discussion

In the Section 4.3, we explored the connections between certain two-dimensional product logics and decidable fragments of first-order logic with counting quantifiers. The decision problems for each of the logics $\mathbf{S5} \times \mathbf{S5}$, $\mathbf{S5} \times \mathbf{Diff}$ and $\mathbf{Diff} \times \mathbf{Diff}$ are known to be coNExpTime-complete; following from the decidability $\mathcal{C}^2$ [96].

However, while $\mathbf{S5} \times \mathbf{S5}$ enjoys the same exponential (square) product fmp as $\mathcal{L}^2$ [51], neither $\mathbf{S5} \times \mathbf{Diff}$ nor $\mathbf{S5} \times \mathbf{Diff}$ can be characterised by their finite *square* product frames. This is a direct analogue of the lack of fmp enjoyed by $\mathcal{C}^2$ [52]. Moreover, we showed in Theorem 6.3, that $\mathbf{Diff} \times \mathbf{Diff}$ cannot be characterised by *any* class of finite frames — square, product or otherwise.

While there have been previous examples of decidable, non-finitely axiomatisable modal logics lacking the finite model property — for example, $\mathbf{K} \times \mathbf{K4.3}$ [38, 77, 72] — the complexity of the decision problem for $\mathbf{Diff} \times \mathbf{Diff}$ is remarkably low for a logic so mired by such infinities.

However, $\mathbf{S5} \times \mathbf{Diff}$ differs markedly from its corresponding first-order fragment; owing to the rectangular nature of product frames, contrasted with the rigid 'squareness' of first-order structures in two-variables. In Section 6.2 we proved that, not only is $\mathbf{S5} \times \mathbf{Diff}$ characterised by its finite product frames, but that every non-theorem $\varphi \notin \mathbf{S5} \times \mathbf{Diff}$ can be refuted in some finite product model whose size does not exceed some *singly* exponential function in the size of φ. In this regard, $\mathbf{S5} \times \mathbf{Diff}$ is more akin to the behaviour of $\mathbf{S5} \times \mathbf{S5}$ than to $\mathbf{Diff} \times \mathbf{Diff}$. These results are summarised in Table 6.1 for comparison.

	$\mathbf{S5} \times \mathbf{S5}$	$\mathbf{S5} \times \mathbf{Diff}$	$\mathbf{Diff} \times \mathbf{Diff}$
abstract fmp	✓	✓	✗
product fmp	✓	✓	✗
square product fmp	✓	✗	✗

Table 6.1: Comparison of finite model properties for products of **S5** and **Diff**.

Moving beyond the immediate connections with first-order logic, it is proved in [118], via a method of *canonical filtration*, that $\mathbf{K} \times \mathbf{Diff}$ possesses the abstract fmp. However this approach yields an upper bound that is non-elementary in the size of the formula.

It is not yet known whether the quasimodel techniques developed here for the treatment of $\mathbf{S5} \times \mathbf{Diff}$ can be adapted to provide an (elementary) upper bound on the size of the *product* models for $\mathbf{K} \times \mathbf{Diff}$. Since the finite frame problem is trivial for product frames of two finite axiomatisable logics, any elementary upper bound on the size of the product models for $\mathbf{K} \times \mathbf{Diff}$ would provide an analogous upper bound on the complexity of its decision problem.

Currently, the only known upper bounds for the complexity of the decision problem for $\mathbf{K} \times \mathbf{Diff}$ are non-elementary. In particular, let **Lin** denote the bimodal *linear temporal logic* with 'future' and 'past' modalities: $\Diamond_F$ and $\Diamond_P$. It is straightforward to check that **Diff** is term-definable within **Lin** via the translation that interprets $\Diamond\varphi := \Diamond_P\varphi \vee \Diamond_F\varphi$ [85].

This translation can be lifted to products, where $\mathbf{K} \times \mathbf{Diff}$ can be defined within the product logic $\mathbf{K} \times \mathbf{Lin}$, whose decision problem is known to be decidable, but not in elementary time [38, 101].

Theorem 6.12. *The decision problem for* $\mathbf{K} \times \mathbf{Diff}$ *is decidable.*

Question 6.13. Does $\mathbf{K} \times \mathbf{Diff}$ have the (effective) product fmp?

Question 6.14. What is the complexity of the decision problem for $\mathbf{K} \times \mathbf{Diff}$?

It is worth noting that the decision problem for $\mathbf{K} \times \mathbf{S5}$ is well-known to be decidable; even coNExpTime-complete [83]. Given the shared coNP-completeness of the decision problems for both **S5** and **Diff**, it is perhaps tempting to conjecture a similar coNExpTime upper bound on the decision problem for $\mathbf{K} \times \mathbf{Diff}$. However, such *prima facie* similarities between products of the form $L \times \mathbf{Diff}$ and those of the form $L \times \mathbf{S5}$, are often deceptive, as will be seen in the chapters to come.

Chapter 7

Computational Complexity of Commutators

The results of Section 5.1 show that — far from being product matching — there is a vast expanse between $[\mathbf{Diff}, \mathbf{Diff}]$ and $\mathbf{Diff} \times \mathbf{Diff}$, with infinitely many logics separating the two. Consequently, the decidability of the decision problem for $[\mathbf{Diff}, \mathbf{Diff}]$ does not follow immediately from that of $\mathbf{Diff} \times \mathbf{Diff}$. Moreover, we noted in Section 6.1 that $[\mathbf{Diff}, \mathbf{Diff}]$ does not even enjoy the *abstract* fmp, and so it is far from obvious that the decision problem for $[\mathbf{Diff}, \mathbf{Diff}]$ would even admit any decision procedure. The current literature provides no tools for dealing with the commutators of logics that are not product matching, and for which the standard techniques of filtration are not applicable.

In this chapter we introduce a novel approach to show that the decision problem for $[\mathbf{Diff}, \mathbf{Diff}]$ is decidable. The technique, described below in Section 7.1, involves reducing the satisfiability problem for $[\mathbf{Diff}, \mathbf{Diff}]$ to that of finding certain appropriately defined quasimodels, similar in approach to the techniques of Section 6.2. However, owing to the lack of any finite model property for $[\mathbf{Diff}, \mathbf{Diff}]$, we are unable to finitize our proposed quasimodels, making exhaustive searches impossible.

Instead, we exploit the rigid structure of the quasimodels, to define a second reduction that reduces the problem of finding an appropriate quasimodel to that of checking satisfiability in the *product* logic $\mathbf{Diff} \times \mathbf{Diff}$. Since the decision problem for $\mathbf{Diff} \times \mathbf{Diff}$ is decidable, these reductions provide us an effective procedure by which we may determine theoremhood in $[\mathbf{Diff}, \mathbf{Diff}]$.

In Section 7.2, we employ a variation on this technique to show that $[\mathbf{S5}, \mathbf{Diff}]$ is characterised by its finite frames. This result exploits the finite product model property of the respective product logic $\mathbf{S5} \times \mathbf{Diff}$, proved above in Section 6.2.

7.1 Decidability of $[\mathbf{Diff}, \mathbf{Diff}]$

We employ a variation of the quasimodel technique seen above, as an intermediary stage in which we reduce the satisfiability problem for $[\mathbf{Diff}, \mathbf{Diff}]$ to that of $\mathbf{Diff} \times \mathbf{Diff}$. Since the decision problem for $\mathbf{Diff} \times \mathbf{Diff}$ is decidable, so it follows that the decision problem for the commutator $[\mathbf{Diff}, \mathbf{Diff}]$ too is decidable.

This novel approach to the treatment of commutators is made possible by the rigidity of the frames for all normal extensions of $[\mathbf{Diff}, \mathbf{Diff}]$, as described in Proposition 5.17. The task of providing upper bound for commutators lacking the finite model property has, hitherto, remained unassailable to the current techniques.

Theorem 7.1. *The decision problem for* $[\mathbf{Diff}, \mathbf{Diff}]$ *is decidable.*

Let $\varphi \in \mathcal{ML}_2$ be an arbitrary bimodal formula of size $n = |\mathrm{sub}(\varphi)|$, and define a *type* for φ to be any Boolean-saturated subset of $\mathrm{sub}(\varphi)$, as in Section 6.2.

Definition 7.2. A *cluster quasistate* for φ is a tuple (T, S_h, S_v, Ω) such that:

(qs1) T is a non-empty set of types for φ, S_h and S_v are binary relations on T, and $\Omega \subseteq \mathrm{sub}(\varphi)$ describes some set of *saturation defects*,

(qs2) For all $t, t' \in T$, we have that tS_h^+t' and tS_v^+t',

(qs3) *(cluster coherence)* For all $t \in T$ and $\Diamond_i\alpha \in \mathrm{sub}(\varphi)$,

$$\exists t' \in T;\ tS_it' \text{ and } \alpha \in t' \qquad \Longrightarrow \qquad \Diamond_i\alpha \in t,$$

for $i = h, v$,

(qs4) *(cluster saturation)* For all $t \in T$ and $\Diamond_i\alpha \in \mathrm{sub}(\varphi)$,

$$\Diamond_i\alpha \in t \text{ and } \Diamond_i\alpha \notin \Omega \qquad \Longrightarrow \qquad \exists t' \in T;\ tS_it' \text{ and } \alpha \in t',$$

for $i = h, v$.

Take Λ to be the set of all possible quasistates for φ, of which there can be at most finitely many. Indeed, it is straightforward to check that:

$$|\Lambda| \leq 2^n \cdot 2^{2^n} \cdot 2^{2^n} \cdot n.$$

Given a quasistate $\boldsymbol{q} = (T^{\boldsymbol{q}}, S_h^{\boldsymbol{q}}, S_v^{\boldsymbol{q}}, \Omega^{\boldsymbol{q}})$, we write $\psi \in \bigcup \boldsymbol{q}$ if there is some $t \in T^{\boldsymbol{q}}$ such that $\psi \in t$, and $\psi \in \bigcap \boldsymbol{q}$ if $\psi \in t$ for all $t \in T^{\boldsymbol{q}}$.

Definition 7.3. A $[\mathbf{Diff}, \mathbf{Diff}]$-*quasimodel* for φ is a tuple (X, Y, λ) such that

(qm1) X and Y are non-empty sets, and λ is a function associating each pair $(x, y) \in X \times Y$ with a cluster quasistate $\lambda(x, y) = (T^{x,y}, S_h^{x,y}, S_v^{x,y}, \Omega^{x,y})$ for φ,

(qm2) There is some $x \in X$ and $y \in Y$ such that $\varphi \in \bigcup \lambda(x, y)$,

(qm3) *($\Diamond_h$-coherence)* For all $x \in X$, $y \in Y$ and $\Diamond_h \alpha \in \mathrm{sub}(\varphi)$,

$$\exists x' \in X;\ x \neq x' \text{ and } \alpha \in \bigcup \lambda(x', y) \qquad \Longrightarrow \qquad \Diamond_h \alpha \in \bigcap \lambda(x, y),$$

(qm4) *($\Diamond_v$-coherence)* For all $x \in X$, $y \in Y$ and $\Diamond_v \alpha \in \mathrm{sub}(\varphi)$,

$$\exists y' \in Y;\ y \neq y' \text{ and } \alpha \in \bigcup \lambda(x, y') \qquad \Longrightarrow \qquad \Diamond_v \alpha \in \bigcap \lambda(x, y),$$

(qm5) *($\Diamond_h$-saturation)* For all $x \in X$, $y \in Y$ and $\Diamond_h \alpha \in \Omega^{x,y}$,

$$\Diamond_h \alpha \in \bigcup \lambda(x, y) \qquad \Longrightarrow \qquad \exists x' \in X;\ x \neq x' \text{ and } \alpha \in \bigcup \lambda(x', y),$$

(qm6) *($\Diamond_v$-saturation)* For all $x \in X$, $y \in Y$ and $\Diamond_v \alpha \in \Omega^{x,y}$,

$$\Diamond_v \alpha \in \bigcup \lambda(x, y) \qquad \Longrightarrow \qquad \exists y' \in Y;\ y \neq y' \text{ and } \alpha \in \bigcup \lambda(x, y'),$$

(qm7) For all $x, x' \in X$ and $y \in Y$,

$$\exists t_0, t_1 \in T^{x,y};\ t_0 S_v^{x,y} t_1 \qquad \Longrightarrow \qquad \exists t_0', t_1' \in T^{x',y};\ t_0' S_v^{x',y} t_1',$$

(qm8) For all $x \in X$ and $y, y' \in Y$,

$$\exists t_0, t_1 \in T^{x,y};\ t_0 S_h^{x,y} t_1 \qquad \Longrightarrow \qquad \exists t_0', t_1' \in T^{x,y'};\ t_0' S_h^{x,y'} t_1'.$$

The following lemmas show that our $[\mathbf{Diff}, \mathbf{Diff}]$-quasimodels contain sufficient information to capture the notion of $[\mathbf{Diff}, \mathbf{Diff}]$-satisfiability.

Lemma 7.4. *If φ is $[\mathbf{Diff}, \mathbf{Diff}]$-satisfiable then there is a $[\mathbf{Diff}, \mathbf{Diff}]$-quasimodel for φ.*

Proof. Suppose that $\mathfrak{M}, r \models \varphi$ for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, where $\mathfrak{F} = (W, R_h, R_v)$ is a rooted frame for $[\mathbf{Diff}, \mathbf{Diff}]$. It then follows from Proposition 5.17 that $\mathfrak{F}$ is described by a grid of clusters. That is to say that there is some (X, Y, h) such that X and Y are non-empty sets and $h : X \times Y \to W^{\sim}$ is a function associating each pair $(x, y) \in X \times Y$ with some cluster of $\mathfrak{F}$, such that if $x \neq x'$ then uR_hv for all $u \in h(x, y)$ and $v \in h(x', y)$ and if $y \neq y'$ then uR_vv for all $u \in h(x, y)$ and $v \in h(x, y')$.

With every $w \in W$ we associate a type

$$\boldsymbol{t}(w) = \{\alpha \in \mathrm{sub}(\varphi) : \mathfrak{M}, w \models \alpha\},$$

and for each $(x, y) \in X \times Y$, we may associate the tuple $\lambda(x, y) = (T^{x,y}, S_h^{x,y}, S_v^{x,y}, \Omega^{x,y})$, where:

- $T^{x,y} = \{\boldsymbol{t}(w) : w \in h(x, y)\}$,
- $\boldsymbol{t}(u)S_i^{x,y}\boldsymbol{t}(v)$ if and only if there is some $u', v' \in h(x, y)$ such that $\boldsymbol{t}(u) = \boldsymbol{t}(u')$, $\boldsymbol{t}(v) = \boldsymbol{t}(v')$ and $u'R_iv'$, for $i = h, v$,
- $\Diamond_i\alpha \in \Omega^{x,y}$ if and only if there is some $u \in h(x, y)$ such that $\Diamond_i\alpha \in \boldsymbol{t}(u)$ and there is no $v \in h(x, y)$ such that $\boldsymbol{t}(u)S_i^{x,y}\boldsymbol{t}(v)$ and $\alpha \in \boldsymbol{t}(v)$, for $i = h, v$.

We claim that $\lambda(x, y)$ is a cluster quasistate for φ satisfying conditions **(qs1)**–**(qs4)**.

- Clearly $T^{x,y}$ is a non-empty set of types, since $h(x, y)$ is non-empty, while it is clear from the definition that $\Omega^{x,y} \subseteq \mathrm{sub}(\varphi)$, as required for **(qs1)**.
- For **(qm2)**, suppose that $\boldsymbol{t}(u), \boldsymbol{t}(v) \in T^{x,y}$ are such that $\boldsymbol{t}(u) \neq \boldsymbol{t}(v)$, for some $u, v \in h(x, y)$. In particular, we have that uR_h^+v and uR_v^+v. It then follows immediately from the definition that $\boldsymbol{t}(u)S_h^{x,y}\boldsymbol{t}(v)$ and $\boldsymbol{t}(u)S_v^{x,y}\boldsymbol{t}(v)$, as required.
- For **(qs3)**, suppose that $\boldsymbol{t}(u), \boldsymbol{t}(v) \in T^{x,y}$ and $\Diamond_i\alpha \in \mathrm{sub}(\varphi)$ are such that $\boldsymbol{t}(u)S_i^{x,y}\boldsymbol{t}(v)$ and $\alpha \in \boldsymbol{t}(v)$. It follows by definition that there is some $u', v' \in h(x, y)$ such that $\boldsymbol{t}(u) = \boldsymbol{t}(u')$, $\boldsymbol{t}(v) = \boldsymbol{t}(v')$ and $u'R_iv'$. Hence we have that $\alpha \in \boldsymbol{t}(v')$ and consequently, that $\Diamond_i\alpha \in \boldsymbol{t}(u') = \boldsymbol{t}(u)$, as required.

– For **(qs4)**, suppose that $\boldsymbol{t}(u) \in T^{x,y}$ and $\Diamond_i\alpha \in \mathrm{sub}(\varphi)$ are such that $\Diamond_i\alpha \in \boldsymbol{t}(u)$ and $\Diamond_i\alpha \notin \Omega^{x,y}$. It is now immediate from the definition of $\Omega^{x,y}$ that there is some $\boldsymbol{t}(v) \in T^{x,y}$ such that $\boldsymbol{t}(u)S_i^X\boldsymbol{t}(v)$ and $\alpha \in \boldsymbol{t}(v)$, as required.

We claim that (X, Y, λ) is a quasimodel for φ satisfying **(qm1)**–**(qm6)**.

– It was demonstrated above that each $\lambda(x, y)$ is a quasistate for φ, for all $x \in X$ and $y \in Y$, as required for **(qm1)**.

– By definition, $\mathfrak{M}, r \models \varphi$, and by definition, there is $x_0 \in X$ and $y_0 \in Y$ such that $r \in h(x_0, y_0)$. Furthermore, we have that $\varphi \in \boldsymbol{t}(r)$ and hence, by construction, $\varphi \in \bigcup\lambda(x_0, y_0)$, as required for **(qm2)**.

– For **(qm3)**, suppose that $x, x' \in X$, $y \in Y$ and $\Diamond_h\alpha \in \mathrm{sub}(\varphi)$ are such that $x \neq x'$ and $\alpha \in \bigcup\lambda(x', y)$, which is to say that $\alpha \in \boldsymbol{t}(w')$ for some $w' \in h(x', y)$. By **(gc1)**, above, wR_hw' for all $w \in h(x, y)$ and $w' \in h(x', y)$. Hence it follows that $\Diamond_h\alpha \in \boldsymbol{t}(w)$ for all $w \in h(x, y)$, which is to say that $\Diamond_h\alpha \in \bigcap\lambda(x, y)$, as required.

– Condition **(qm4)** is analogous.

– For **(qm5)**, suppose that $x \in X$, $y \in Y$ and $\Diamond_h\alpha \in \mathrm{sub}(\varphi)$ are such that $\Diamond_h\alpha \in \bigcup\Omega^{x,y}$. By definition, there is some $u \in h(x, y)$ such that $\Diamond_h\alpha \in \boldsymbol{t}(u)$ and there is no $v \in h(x, y)$ such that $\boldsymbol{t}(u)S_i^{x,y}\boldsymbol{t}(v)$ and $\alpha \in \boldsymbol{t}(v)$. However, there must be some $v' \in W$ such that uR_hv' and $\alpha \in \boldsymbol{t}(v')$. It then follows that there is some $x' \in X$ such that $x \neq x'$ and $v' \in h(x', y)$. Moreover, we have by definition that $\alpha \in \bigcup\lambda(x', y)$, as required.

– Condition **(qm6)** is analogous.

– For **(qm7)**, suppose that $x, x' \in X$ and $y \in Y$ are such that $tS_v^{x,y}t'$ for some $t, t' \in T^{x,y}$. It follows that there must be some $u, v \in h(x, y)$ such that $\boldsymbol{t}(u) = t$ and $\boldsymbol{t}(v) = t'$, in particular we have that uR_vv. Let $u' \in h(x', y)$ then by **(gc2)**, we have that uR_hu'. Hence by (*chr*), there is some $v' \in W$ such that $u'R_vv'$ and vR_hv'. Moreover, since R_h^+ is an equivalence relation, we have that uR_h^+v', and thus $v' \in h(x', y)$. Hence there are $\boldsymbol{t}(u'), \boldsymbol{t}(v') \in T^{x',y}$ such that $\boldsymbol{t}(u')S_v^{x',y}\boldsymbol{t}(v')$, as required.

– Condition **(qm8)** is analogous.

Hence it follows that (X, Y, λ) is an appropriate quasimodel for φ, as required. □

Conversely, we show that, from any [**Diff**, **Diff**]-quasimodel for φ, we can effectively construct a model for φ based on some frame for [**Diff**, **Diff**].

Lemma 7.5. *If φ has a* $[\mathbf{Diff}, \mathbf{Diff}]$*-quasimodel then φ is* $[\mathbf{Diff}, \mathbf{Diff}]$*-satisfiable.*

Proof. Suppose that (X, Y, λ) is a quasimodel for φ. We define a new model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, where $\mathfrak{F} = (W, R_h, R_v)$ by taking

$$W = \{(x, y, t) : x \in X, y \in Y \text{ and } t \in \lambda(x, y)\},$$

and for all $(x, y, t), (x', y', t') \in W$,

$$(x, y, t)R_h(x', y', t') \iff y = y' \quad \text{and} \quad \big(x \neq x' \text{ or } tS_h^{x,y}t'\big),$$

$$(x, y, t)R_v(x', y', t') \iff x = x' \quad \text{and} \quad \big(y \neq y' \text{ or } tS_v^{x,y}t'\big).$$

For each propositional variable $p \in \mathrm{sub}(\varphi)$, we take

$$\mathfrak{V}(p) = \{(x, y, t) \in W \ : \ p \in t\}.$$

It follows from **(qs2)** that each quasistate $\lambda(x, y)$ is a frame for $[\mathbf{Diff}, \mathbf{Diff}]$, for $x \in X$ and $y \in Y$. By **(qm7)**, any vertical S_v-transition *within* a quasistate can be mirrored by a similar S_v-transition in every *horizontal* neighbouring quasistate. Analogously, by **(qm8)**, any horizontal S_h-transition within a quasistate can be mirrored by an S_h-transition in every *vertically* neighbouring quasistate. It therefore follows that $\mathfrak{F}$ validates each of the axioms (com^l), (com^r) and (chr). From which it follows that $\mathfrak{F}$ is a frame for $[\mathbf{Diff}, \mathbf{Diff}]$, as required. It remains to show that $\mathfrak{M}$ is a model for φ.

We claim that for all $(x, y, t) \in W$ and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}, (x, y, t) \models \psi \iff \psi \in t. \tag{I.H.}$$

The cases where ψ is a propositional variable or a Boolean combination of smaller formulas are trivial and follow immediately from the definitions.
So suppose that $\mathfrak{M}, (x, y, t) \models \Diamond_h\alpha$, for some $\alpha \in \mathrm{sub}(\varphi)$. It follows that there is some $(x', y', t') \in W$ such that $(x, y, t)R_h(x', y', t')$ and $\mathfrak{M}, (x', y', t') \models \alpha$. By the induction hypothesis we find that $\alpha \in t'$, while by the definition of R_h, we have that $y = y'$ and either $x \neq x'$ or $tS_h^{x,y}t'$. We have two cases to consider:

- If $x \neq x'$ then by **(qm3)** we have that $\Diamond_h\alpha \in \bigcap \lambda(x, y)$, and thus $\Diamond_h\alpha \in t$.

– Otherwise $tS_h^{x,y}t'$ and it follows from **(qs3)** that $\Diamond_h\alpha \in t$.

Conversely, suppose that $\Diamond_h\alpha \in t$. Then we have two cases to consider, depending on whether or not $\Diamond_h\alpha$ is belongs to $\Omega^{x,y}$:

– If $\Diamond_h\alpha \notin \Omega^{x,y}$ then by definition there is some $t' \in T^{x,y}$ such that $tS_h^{x,y}t'$ and $\alpha \in t'$. Moreover it follows from the definition that $(x, y, t)R_h(x, y, t')$.

– If $\Diamond_h\alpha \in \Omega^{x,y}$ then by **(qm5)** there is some $x' \in W_h$ and $t' \in T^{x',y}$ such that $x \neq x'$ and $\alpha \in t'$. Moreover it follows from the definition that $(x, y, t)R_h(x', y, t')$.

In both cases we find that there is some $x' \in X$ and $t' \in T^{x',y}$ such that $(x, y, t)R_h(x', y, t')$ and $\alpha \in t'$. By the induction hypothesis, we have that $\mathfrak{M}, (x', y, t') \models \alpha$, from which it follows that $\mathfrak{M}, (x, y, t) \models \Diamond_h\alpha$.

The case where ψ is of the form $\Diamond_v\alpha$, for some $\alpha \in \mathrm{sub}(\varphi)$, is analogous. Hence we have that $\mathfrak{M}, (x, y, t) \models \psi$ if and only if $\psi \in t$, for all $(x, y, t) \in W$ and $\psi \in \mathrm{sub}(\varphi)$. In particular, it follows from **(qm2)** that there is some $(x_0, y_0, t_0) \in W$ such that $\mathfrak{M}, (x_0, y_0, t_0) \models \varphi$, as required. $\square$

Thus we have reduced the problem of deciding whether a φ is $[\mathbf{Diff}, \mathbf{Diff}]$-satisfiable to that of checking whether φ has a suitable quasimodel. This exercise is fruitless, however, unless we have some means by which we can effectively search for quasimodels.

Fortunately, owing to the rigid grid-like structure of our quasimodels for φ, we may further reduce the problem of checking whether φ has a quasimodel to that of satisfiability in the *product logic* $\mathbf{Diff} \times \mathbf{Diff}$.

Firstly, we associate with each quasistate $\boldsymbol{q} \in \Lambda$ some propositional variable $\widetilde{\boldsymbol{q}} \in \textsc{prop}$. We then define for each $\psi \in \mathrm{sub}(\varphi)$, the following abbreviations

$$
\begin{aligned}
[\psi]^\exists &:= \bigvee \{\widetilde{\boldsymbol{q}} : \boldsymbol{q} \in \Lambda \text{ and } \psi \in \textstyle\bigcup \boldsymbol{q}\}\\
[\psi]^\forall &:= \bigvee \{\widetilde{\boldsymbol{q}} : \boldsymbol{q} \in \Lambda \text{ and } \psi \in \textstyle\bigcap \boldsymbol{q}\},\\
[\psi]^\Omega &:= \bigvee \{\widetilde{\boldsymbol{q}} : \boldsymbol{q} \in \Lambda \text{ and } \psi \in \Omega^{\boldsymbol{q}}\}.
\end{aligned}
$$

Furthermore we define

$$\theta_i \;:=\; \bigvee\{\widetilde{\boldsymbol{q}} \;:\; \exists t, t' \in T^{\boldsymbol{q}} \text{ such that } tS_i^{\boldsymbol{q}}t'\}$$

for $i = h, v$.
Take qm_φ to be the conjunction of the following formulas:

$$\Box_h^+\Box_v^+ \bigvee_{\boldsymbol{q}\in\Lambda} \widetilde{\boldsymbol{q}} \wedge \Box_h^+\Box_v^+ \bigwedge_{\boldsymbol{q}\neq\boldsymbol{q}'} \neg(\widetilde{\boldsymbol{q}} \wedge \widetilde{\boldsymbol{q}'}) \wedge \Diamond_h^+\Diamond_v^+[\varphi]^\exists, \tag{7.1}$$

$$\Box_h^+\Box_v^+ \bigwedge_{\Diamond_i\alpha\in\mathrm{sub}(\varphi)} (\Diamond_i[\alpha]^\exists \to [\Diamond_i\alpha]^\forall), \qquad \text{for } i = h, v, \tag{7.2}$$

$$\Box_h^+\Box_v^+ \bigwedge_{\Diamond_i\alpha\in\mathrm{sub}(\varphi)} ([\Diamond_i\alpha]^\exists \wedge [\Diamond_i\alpha]^\Omega \to \Diamond_i[\alpha]^\exists), \qquad \text{for } i = h, v, \tag{7.3}$$

$$\Box_h^+\Box_v^+(\theta_h \to \Box_v\theta_h) \wedge \Box_h^+\Box_v^+(\theta_v \to \Box_h\theta_v). \tag{7.4}$$

The intuition behind these formulas is explained by way of the following lemmas.

Lemma 7.6. *There is a* $[\mathbf{Diff}, \mathbf{Diff}]$*-quasimodel for* φ *if and only if* qm_φ *is* $\mathbf{Diff} \times \mathbf{Diff}$*-satisfiable.*

Proof. ($\Rightarrow$) Suppose that (X, Y, λ) is a quasimodel for φ. Let $\mathfrak{F}_h = (X, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$ and $\mathfrak{F}_v = (Y, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, and define a new model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$ over $\mathfrak{F}_h \times \mathfrak{F}_v$, by taking

$$(x, y) \in \mathfrak{V}(\widetilde{\boldsymbol{q}}) \qquad \Longleftrightarrow \qquad \lambda(x, y) = \boldsymbol{q},$$

for all $x \in X$ $y \in Y$, and all quasistates $\boldsymbol{q} \in \Lambda$. The following are then immediate consequences of the definitions:

$$\begin{aligned}
\mathfrak{M}, (x, y) \models [\psi]^\exists &\iff \psi \in \bigcup \lambda(x, y),\\
\mathfrak{M}, (x, y) \models [\psi]^\forall &\iff \psi \in \bigcap \lambda(x, y),\\
\mathfrak{M}, (x, y) \models [\psi]^\Omega &\iff \psi \in \Omega^{x,y},\\
\mathfrak{M}, (x, y) \models \theta_h &\iff \exists t, t' \in T^{x,y};\ tS_h^{x,y}t',\\
\mathfrak{M}, (x, y) \models \theta_v &\iff \exists t, t' \in T^{x,y};\ tS_v^{x,y}t'.
\end{aligned}$$

It then follows that each of the conjuncts (7.1)–(7.4) are simply reformulations of conditions **(qm2)** – **(qm8)**. Hence we must have that qm_φ is $\mathbf{Diff} \times \mathbf{Diff}$-satisfiable, as required.

($\Leftarrow$) Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{qm}_\varphi$, for some product model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_i = (W_i, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, for $i = h, v$.

We define a quasimodel (W_h, W_v, λ) by taking

$$\lambda(x, y) = \boldsymbol{q} \qquad \Longleftrightarrow \qquad (x, y) \in \mathfrak{V}(\widetilde{\boldsymbol{q}}),$$

for all quasistates $\boldsymbol{q} \in \Lambda$.

By (7.1), we are assured that λ is well-defined, and that there is some $x \in W_h$, $y \in W_v$ such that $\varphi \in \bigcup \lambda(x, y)$, as required for **(qm2)**. Conditions **(qm3)**–**(qm4)** are satisfied by (7.2), while conditions **(qm5)**–**(qm6)** are satisfied by (7.3). Lastly, (7.4) ensures that conditions **(qm7)**–**(qm8)** are satisfied. Hence (W_h, W_v, λ) is a quasimodel for φ as required.

□

Hence, it follows from Lemmas 7.4–7.6 that φ is $[\mathbf{Diff}, \mathbf{Diff}]$-satisfiable if and only if qm_φ is $\mathbf{Diff} \times \mathbf{Diff}$-satisfiable. It then follows immediately from Theorem 4.3 that the satisfiability problem — and, thus, the decision problem — for $[\mathbf{Diff}, \mathbf{Diff}]$ is decidable, thereby completing the proof of Theorem 7.1.

Furthermore, it follows from Theorem 3.8 that both $[\mathbf{S5}, \mathbf{Diff}]$ and $[\mathbf{Diff}, \mathbf{Diff}]$ are Kripke complete, and that (W, R_h, R_v) is a frame for $[\mathbf{Diff}, \mathbf{Diff}]$ if and only if (W, R_h^+, R_v) is a frame for $[\mathbf{S5}, \mathbf{Diff}]$. It follows that $[\mathbf{S5}, \mathbf{Diff}]$ is term-definable within $[\mathbf{Diff}, \mathbf{Diff}]$, via the translation that maps $\Diamond_h \varphi \mapsto \varphi \vee \Diamond_h \varphi$. Thus we may employ the results of Theorem 7.1, to provide a similar upper bound on the complexity of the decision problem for $[\mathbf{S5}, \mathbf{Diff}]$.

Corollary 7.7. *The decision problem for* $[\mathbf{S5}, \mathbf{Diff}]$ *is decidable.*

Note, however, that the above reduction from commutators to products incurs a *triple-exponential* increase in complexity, owing to the multitude of possible cluster quasistates, which greatly inflate the size of qm_φ relative to φ. Thus this procedure informs us of only a coN4ExpTime upper bound on the complexity of the decision problems for both

$[\mathbf{Diff}, \mathbf{Diff}]$ and $[\mathbf{S5}, \mathbf{Diff}]$. While these upper bounds are reassuringly elementary, it is reasonable to suggest that they may remain a far-cry from optimality.

Question 7.8. What is the computational complexity of the decision problems for both $[\mathbf{S5}, \mathbf{Diff}]$ and $[\mathbf{Diff}, \mathbf{Diff}]$?

7.2 Finite Model Property of $[\mathbf{S5}, \mathbf{Diff}]$

In [39] the authors prove, via a method of filtration, that $L \times \mathbf{S5} = [L, \mathbf{S5}]$ has the double-exponential fmp, whenever L is one of the logics $\mathbf{K}, \mathbf{T}, \mathbf{D}, \mathbf{K4}, \mathbf{S4}$ or $\mathbf{S5}$. However it is not clear that the techniques employed there can be easily adapted to provide a similar upper bound for $[\mathbf{S5}, \mathbf{Diff}]$; clearly they are of no consequence to $[\mathbf{Diff}, \mathbf{Diff}]$, which lacks even the abstract fmp (see Theorem 6.3).

However, by considering a variation on the above quasimodel construction we can exploit the finite product model property of $\mathbf{S5} \times \mathbf{Diff}$, to provide an upper bound on the size of the quasimodels for $[\mathbf{S5}, \mathbf{Diff}]$, thus defined.

Definition 7.9. An $[\mathbf{S5}, \mathbf{Diff}]$-*quasimodel* for φ is a tuple (X, Y, λ) satisfying conditions **(qm1)**, **(qm2)**, **(qm4)**, **(qm6)** – **(qm8)** of Definition 7.3, together with the following, **S5**-specific criteria:

(qm3′) *($\Diamond_h$-coherence)* For all $x \in X$, $y \in Y$ and $\Diamond_h\alpha \in \mathrm{sub}(\varphi)$,

$$\exists x' \in X;\ \alpha \in \bigcup \lambda(x', y) \qquad \Longrightarrow \qquad \Diamond_h\alpha \in \bigcap \lambda(x, y),$$

(qm5′) *($\Diamond_h$-saturation)* For all $x \in X$, $y \in Y$ and $\Diamond_h\alpha \in \Omega^{x,y}$,

$$\Diamond_h\alpha \in \bigcup \lambda(x, y) \qquad \Longrightarrow \qquad \exists x' \in X;\ \alpha \in \bigcup \lambda(x', y),$$

(qm9) *($\Diamond_h$-reflexivity)* For all $x \in X$ and $y \in Y$, we have that $tS_h^{x,y}t$, for all $t \in T^{x,y}$.

It is then readily proved, in the style of Lemmas 7.4 – 7.5, that φ is $[\mathbf{S5}, \mathbf{Diff}]$-satisfiable if and only if φ has a $[\mathbf{S5}, \mathbf{Diff}]$-quasimodel.

Lemma 7.10. *φ is $[\mathbf{S5}, \mathbf{Diff}]$-satisfiable if and only if φ has a $[\mathbf{S5}, \mathbf{Diff}]$-quasimodel*

Proof. Analogous to that of Lemmas 7.4 and 7.5, taken together. $\square$

Take $\mathsf{qm}_\varphi^{\mathsf{ref}}$ to be the conjunction of qm_φ together with the following formula:

$$\Box_h \Box_v^+ \bigvee \{\widetilde{\boldsymbol{q}} : \boldsymbol{q} \in \Lambda \text{ and } tS_h t \text{ for all } t \in T\}, \tag{7.5}$$

which stipulates that only those quasistates comprising only S_h-reflexive elements are to be permitted. This affords us the following natural analogue of Lemma 7.6.

Lemma 7.11. *There is an $[\mathbf{S5}, \mathbf{Diff}]$-quasimodel for φ if and only if $\mathsf{qm}_\varphi^{\mathsf{ref}}$ is $\mathbf{S5} \times \mathbf{Diff}$-satisfiable.*

Proof. The proof is analogous to that of Lemma 7.6, with the addition that condition **(qm9)** is captured by (7.5). $\square$

Furthermore, we may note from the above construction that φ is satisfiable in a finite frame for $[\mathbf{S5}, \mathbf{Diff}]$, whenever $\mathsf{qm}_\varphi^{\mathsf{ref}}$ is satisfiable in a finite product frame for $\mathbf{S5} \times \mathbf{Diff}$. Hence it follows from the exponential product fmp for $\mathbf{S5} \times \mathbf{Diff}$, proved in Theorem 6.6, that every non-theorem of $[\mathbf{S5}, \mathbf{Diff}]$ can be refuted in a frame that can be described by a finite grid of clusters. Thus we may conclude that $[\mathbf{S5}, \mathbf{Diff}]$ has the abstract fmp.

Theorem 7.12. $[\mathbf{S5}, \mathbf{Diff}]$ *has the abstract fmp.*

This $[\mathbf{S5}, \mathbf{Diff}]$-quasimodel variation fairs no better than its precursor, and incurs a similar triple-exponential increase in the size of the quasimodel. Moreover, since each cluster may contain exponentially many types, this provides us a prodigious quadruple-exponential fmp for $[\mathbf{S5}, \mathbf{Diff}]$.

7.3 Discussion

The explosion of complexity incurred by our quasimodel approach stems from the vast multitude of possible quasistates, as we have defined them here. However, upon more careful inspection, we may note that not all such quasistates are necessary for our reduction. It is, therefore, possible that a more stringent definition of what counts as a quasistates may serve to curtail this explosion of complexity. It is perhaps reasonable to conjecture that such a reduction in the number of possible quasistates could provide a coN2ExpTime upper bound on the decision problems for both $[\mathbf{Diff}, \mathbf{Diff}]$ and $[\mathbf{S5}, \mathbf{Diff}]$.

Question 7.13. What is the computational complexity of the decision problems for both $[\mathbf{Diff}, \mathbf{Diff}]$ and $[\mathbf{S5}, \mathbf{Diff}]$?

Question 7.14. What is the optimal bound on the size of the models for $[\mathbf{S5}, \mathbf{Diff}]$?

It is not known whether this strategy can be employed to ascertain the computational complexity of other non product matching logics such as **K4.3** and **K**, or **K4.3** and **S5**, whose products are known to be decidable [38].

Question 7.15. Is the decision problem for either $[\mathbf{K4.3}, \mathbf{K}]$ or $[\mathbf{K4.3}, \mathbf{S5}]$ decidable?

Chapter 8

Computational Complexity of Products

In this chapter we examine the computational complexity of the decision problem for logics of the form $L \times \mathbf{Diff}$, for a variety of Kripke complete unimodal and bimodal logics L. By definition, product frames are always 'grid-like', and so a typical approach to proving lower bounds is to encode complex grid-based problems such as tiling problems or Turing machine reachability problems. With both 'next-time' and 'universal' modalities in both dimensions, such reductions are relatively straightforward — a classic example is given by the undecidable decision problem for $\mathbf{K}_u \times \mathbf{K}_u$, considered in [38, Theorem 5.37]. However, in lieu of an appropriate 'next-time' operator, it is sometimes possible to employ a version of Cantor's enumeration of the $\omega \times \omega$-plane to encode a grid-like structure along an ascending sequence of 'diagonal' points, with pointers emulating the required horizontal and vertical 'next-time' operators [41, 84, 102].

Owing to the lack of structure endemic in the vertical component of those frames for $L \times \mathbf{Diff}$, these tricks fail to find obvious application. In particular, while an ascending sequence of diagonal points may be definable in our frames, the lack of directionality in the vertical component prevents any accurate encoding of the requisite 'next-time' pointers.

In this chapter we introduce a novel technique, whereby we may directly exploit the grid-like structure of the product frames to encode various (Minsky) counter machine problems, thereby obtaining undecidable lower bounds for a range of products of the form $L \times \mathbf{Diff}$. The use of counter machines is attractive, as it appears to require far less structure than is required of those proofs involving the aforementioned grid-based problems.

We compare their complexity with the relatively modest complexity of the decision problem for those corresponding logics of the form $L \times \mathbf{S5}$, which are typically decidable whenever L is decidable, and those corresponding logics of the form $L \times \mathbf{K4.3}$, which are typically undecidable even when L is decidable [38]. We show that, despite the *prima facie* similarities between **S5** and **Diff**, the computational complexity of those products of the form $L \times \mathbf{Diff}$ can often vastly exceed that of their $L \times \mathbf{S5}$ counterparts, and are — in this regard — more akin to their $L \times \mathbf{K4.3}$ counterparts.

In Section 8.1, we introduce (Minsky) counter machines together with a description of the various reachability problems that shall be employed throughout this chapter, and their respective computational complexities.

In Section 8.2, we introduce this technique for those cases in which we still have both horizontal 'universal' and 'next-time' operators at our disposal. The results of this section are published in [58]. In Section 8.3 we sharpen this technique and discuss products of the form $L \times \mathbf{Diff}$, where L is characterised by some class of *linear orders*, without a next-time operator. The results of this section are published in [59].

Finally, in Section 8.4, we provide a polynomial reduction from the decision problem for $L \times \mathbf{Diff}$ to that of $L \times \mathbf{K4.3}$, whenever L is Kripke complete, thereby generalising many pre-existing undecidability results obtained in [84, 102].

8.1 Counter Machines

Counter machines were introduced by Minsky in [87] as an alternative Turing-complete model of computation that more closely resembles that of modern digital computers than that of Turing machines [126].

Formally, a *counter machine (CM)* is a 5-tuple $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$, where Q is a finite set of control states, among which is the initial state $q_{\mathsf{init}} \in Q$. The number of available counters is specified by $n < \omega$, and $\Delta \subseteq Q \times Op_n \times Q$ describes a set of labelled transitions over Q, whose labels are taken from the set Op_n, comprising the following permissible *counter operations*, for $i < n$:

- i^{++} (increment counter i by one),
- i^{--} (decrement counter i by one),
- $i^{??}$ (test whether counter i is empty).

Lastly $H \subseteq Q$ denotes a set of *halting states*, fully determined by Δ, taking $q \in H$ if and only if there is no transition $(q, \alpha, q') \in \Delta$, for any $q' \in Q$ and $\alpha \in Op_n$.

The *configurations* of $\mathcal{M}$ are those tuples $(q, v) \in Q \times \omega^n$, where $q \in Q$ denotes the internal state of $\mathcal{M}$ and $v : n \to \omega$ is a function describing the current values held by each of the n counters. The set of all configurations of $\mathcal{M}$ is denoted $\mathrm{Conf}_{\mathcal{M}}$.

Two configurations are said to be $\mathcal{M}$*-consecutive*, written $(q, v) \xrightarrow{\mathcal{M}} (q', v')$, if there is some $\alpha \in Op_n$ such that $(q, \alpha, q') \in \Delta$ and, for all $i < n$,

– If $\alpha = i^{++}$ then $v'(i) = v(i) + 1$,

– If $\alpha = i^{--}$ then $v'(i) = v(i) - 1$,

– If $\alpha = i^{??}$ then $v'(i) = v(i) = 0$,

– If $\alpha \in \{j^{++}, j^{--}, j^{??}\}$ for some $j \neq i$, then $v'(j) = v(j)$.

A *(reliable) computation* of $\mathcal{M}$ is a sequence of $\mathcal{M}$-consecutive configurations $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L\rangle$ of length $L \leq \omega$, such that:

– $q_0 = q_{\mathsf{init}}$ and $v_0 = \vec{0}$, where $\vec{0}$ denotes the function that assigns zero to all counters;

– If $k > 0$ then $(q_{k-1}, v_{k-1}) \xrightarrow{\mathcal{M}} (q_k, v_k)$,

– $q_k \in H$ if and only if $k + 1 = L$.

for all $k < L$. We say that a computation is a *terminating* if its length $L < \omega$ is finite.

Note that we are assuming here that all our computations are initialised with empty counters. However, this poses no loss of generality, for if we desire our computations to be initialised at some configuration other than $(q_{\mathsf{init}}, \vec{0})$, we may construct a new machine that 'mimics' the instructions of $\mathcal{M}$, after first 'loading' the prescribed counter values. There is a one-to-one correspondence between the computations of $\mathcal{M}$ initialised with this newly prescribed configuration, and the computation of this new machine initialised with empty counters.

For our purposes, we will be interested in the following decision problems that may be asked of counter machines, whose complexity is well-established.

<u>CM TERMINATION:</u> (Σ^0_1-complete [87])
Given a counter machine $\mathcal{M}$, does every reliable computation of $\mathcal{M}$ eventually terminate?

CM REACHABILITY: (Σ^0_1-complete [87])

Given a counter machine $\mathcal{M}$, and a state $\ell \in Q - H$, does $\mathcal{M}$ have a reliable computation in which ℓ occurs?

CM BÜCHI: (Σ^1_1-complete [5])

Given a counter machine $\mathcal{M}$, and a state $\ell \in Q - H$, does $\mathcal{M}$ have a reliable computation in which ℓ occurs *infinitely often*?

8.2 Products with 'Next-time'

In this section we introduce this technique for cases where we still have both horizontal 'universal' and 'next-time' operators at our disposal. In particular, we consider the cases where the horizontal component in question is the logic **K** augmented by either a *universal modality* or a *master (transitive closure) modality.* In both cases, it is the modal operator native to **K** that acts as the 'next-time' operator, while the rôle of the universal operator, is played by the secondary modality.

8.2.1 Universal Modality

Given a Kripke complete unimodal logic L, we define L_u to be the bimodal logic, having modal operators $\Diamond$ and $\Diamond^u$, characterised by all those frames of the form (W, R, W^2), where (W, R) is a frame for L. That is to say that $\Diamond^u$ is an additional *universal modality* capable of expressing the truth of a given formula anywhere in the universe W, irrespective of the primary accessibility relation R. Modalities of this kind were introduced and investigated by Goranko and Passy [50].

Of course, by Proposition 2.4, the intended 'universality' of $\Diamond^u$ is not modally expressible by any modal axioms, and so L_u admits many frames in addition to those of the form (W, R, W^2). However, since the universal relation on W is an equivalence relation containing R — a property that can be expressed by modal axioms — so too must be the relation interpreting $\Diamond^u$ in any frame for L_u.

As may be expected, the addition of such a universal modality often leads to a considerable increase in computational complexity, as was demonstrated by Hemaspaandra [121, 60]. For example, while the decision problem for **K** is PSPACE-complete [78], the decision problem for $\mathbf{K}_u$ is EXPTIME-complete [60].

Moreover, while the decision problem for $\mathbf{K}\times\mathbf{K}$ is known to be coNExpTime-complete, $\mathbf{K}_u \times \mathbf{K}_u$ provides a classic illustration of a logic whose undecidability follows from a straightforward encoding of the *unconstrained tiling problem*† [38].

It was proved in [83] that $\mathbf{K} \times \mathbf{S5}$ is coNExpTime-complete, while the decidability of $\mathbf{K}_u \times \mathbf{S5}$ follows from the coN2ExpTime upper bound of $\mathbf{CPDL} \times \mathbf{S5}$ [105], where $\mathbf{CPDL}$ denotes *propositional dynamic logic* with the addition of *converse* actions (see [38] for definitions and references). This is to say that, while the transition from $\mathbf{K}\times\mathbf{S5}$ to $\mathbf{K}_u\times\mathbf{S5}$ marks a considerable increase in complexity, the decision problem remains decidable.

In this section we prove that, unlike $\mathbf{K} \times \mathbf{S5}$, which remains decidable when augmented by a horizontal universal modality, there is a jump in complexity from the decidability of $\mathbf{K} \times \mathbf{Diff}$, discussed in Section 6.3, to the undecidability of $\mathbf{K}_u \times \mathbf{Diff}$.

Theorem 8.1 (Hampson-Kurucz [58]). *Let $\mathcal{C}$ be any class of bimodal frames for $\mathbf{K}_u$ such that $(\omega, S, \omega^2) \in \mathcal{C}$, where S is the successor relation‡ on ω.*
Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Σ^0_1-hard.

We fix an arbitrary counter machine $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$. Our goal will be to construct a $\mathcal{ML}_3$-formula $\psi_{\mathcal{M}}$ that is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if $\mathcal{M}$ has a *non*-terminating computation. Since the Termination problem for reliable counter machines is Σ^0_1-hard, so too must be the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$.

To this end, take $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$ to be an arbitrary product model, such that $\mathfrak{F}_h = (W_h, R_h, R_h') \in \mathcal{C}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

For each $i < n$, let p_i be a fresh propositional variable, and define

$$\Sigma_i(x) = \{y \in W_v : r_v R_v^+ y \text{ and } \mathfrak{M}, (x, y) \models p_i\},$$

for all $x \in W_h$. The intention here, is to represent, in the horizontal dimension, the evolving instances of time, while in the vertical dimension we represent the value of counter i, at time $x \in W_h$, by the cardinality of $\Sigma_i(x)$.

†The *unconstrained tiling problem* asks whether a given set of tiles types may be compatibly tesselated to cover the entire $(\omega \times \omega)$ plane. The undecidability of this problem was originally proved by Berger [13], the proof of which was subsequently simplified by Robinson [103]. See [129] and references therein, for details.

‡That is to say $(x, y) \in S$ if and only if $y = x + 1$.

To govern the behaviour of these *'counter variables'* we introduce, for each $i < n$, the following formulas:

$$\mathsf{fix}_i := \Box_v^+(\Diamond_h p_i \to p_i) \wedge \Box_v^+(p_i \to \Box_h p_i), \tag{8.1}$$

$$\mathsf{inc}_i := \Diamond_v^{=1}(\neg p_i \wedge \Diamond_h p_i) \wedge \Box_v^+(p_i \to \Box_h p_i) \wedge \Box_v^+(\Diamond_h p_i \to \Box_h p_i), \tag{8.2}$$

$$\mathsf{dec}_i := \Diamond_v^{=1}(p_i \wedge \Diamond_h \neg p_i) \wedge \Box_v^+(\Diamond_h p_i \to p_i) \wedge \Box_v^+(\Diamond_h p_i \to \Box_h p_i). \tag{8.3}$$

The interpretation of these formulas is best explained by way of the following lemma.

Lemma 8.2 (Counting Lemma)**.** *Let $x, x' \in W_h$ be such that xR_hx'.*

(i) *If $\mathfrak{M}, (x, r_v) \models \mathsf{fix}_i$, then $\Sigma_i(x') = \Sigma_i(x)$,*

(ii) *If $\mathfrak{M}, (x, r_v) \models \mathsf{inc}_i$, then $\Sigma_i(x') = \Sigma_i(x) \cup \{z\}$ for some $z \notin \Sigma_i(x)$,*

(iii) *If $\mathfrak{M}, (x, r_v) \models \mathsf{dec}_i$, then $\Sigma_i(x') = \Sigma_i(x) - \{z\}$ for some $z \in \Sigma_i(x)$.*

Proof. (i) Suppose $y \in \Sigma_i(x')$ then by definition $r_vR_v^+y$ and $\mathfrak{M}, (x', y) \models p_i$. Hence we have that $\mathfrak{M}, (x, y) \models \Diamond_h p_i$, since xR_hx'. Therefore $\mathfrak{M}, (x, y) \models p_i$, by (8.1), which is to say that $y \in \Sigma_i(x)$.

Conversely, suppose $y \in \Sigma_i(x)$ then by definition $r_vR_v^+y$ and $\mathfrak{M}, (x, y) \models p_i$. Hence we have that $\mathfrak{M}, (x, y) \models \Box_h p_i$, by (8.1). Therefore $\mathfrak{M}, (x', y) \models p_i$, since xR_hx', which is to say that $y \in \Sigma_i(x')$.

(ii) By (8.2) there is some unique $z \in W_v$ such that $r_vR_v^+y$ and $\mathfrak{M}, (x, z) \models \neg p_i \wedge \Diamond_h p_i$. In particular, we have that $z \notin \Sigma_i(x)$.

Now suppose $y \in \Sigma_i(x')$ then by definition $r_vR_v^+y$ and $(x', y) \models p_i$. Hence we have that $\mathfrak{M}, (x, y) \models \Diamond_h p_i$, since xR_hx'. It then follows that, either $y = z$ or $\mathfrak{M}, (x, y) \models p_i$, which is to say that $y \in \Sigma_i(x) \cup \{z\}$.

Conversely, suppose $y \in \Sigma_i(x)$ then by definition $r_vR_v^+y$ and $\mathfrak{M}, (x, y) \models p_i$. Hence we have that $\mathfrak{M}, (x, y) \models \Box_h p_i$, by (8.2). Therefore $\mathfrak{M}, (x', y) \models p_i$, since xR_hx', which is to say that $y \in \Sigma_i(x')$.

Furthermore we have $\mathfrak{M}, (x, z) \models \Diamond_h p_i$, and so by (8.2), $\mathfrak{M}, (x, z) \models \Box_h p_i$. Hence $\mathfrak{M}, (x', z) \models p_i$, since xR_hx', which is to say that $z \in \Sigma_i(x')$.

(iii) By (8.3) there is some unique $z \in W_v$ such that $\mathfrak{M}, (x, z) \models p_i \wedge \Diamond_h \neg p_i$. In particular, we have that $z \in \Sigma_i(x)$.

Now suppose $y \in \Sigma_i(x')$. Then by definition $r_v R_v^+ y$ and $(x', y) \models p_i$. Hence we have that $\mathfrak{M}, (x, y) \models \Diamond_h p_i$, since $x R_h x'$. Therefore $\mathfrak{M}, (x, y) \models p_i$, by (8.3), which is to say that $y \in \Sigma_i(x)$. Moreover, by (8.3), we have that $\mathfrak{M}, (x, y) \models \Box_h p_i$ and hence $y \neq z$, since $\mathfrak{M}, (x, z) \models \Diamond_h \neg p_i$.

Conversely, suppose $y \in \Sigma_i(x)$ and $y \neq z$. Then by definition $r_v R_v^+ y$, $\mathfrak{M}, (x, y) \models p_i$, and $\mathfrak{M}, (x, y) \models \Box_h p_i$, since $y \neq z$. Therefore $\mathfrak{M}, (x', y) \models p_i$, since $x R_h x'$, which is to say that $y \in \Sigma_i(x')$.

□

We then specify the action of each counter operation $\alpha \in Op_n$ by the following combination of these atomic actions on the individual counter variables:

$$\mathsf{Do}_\alpha \;:=\; \begin{cases} \mathsf{inc}_i \wedge \bigwedge_{j \neq i} \mathsf{fix}_j & \text{if } \alpha = i^{++}, \\ \mathsf{dec}_i \wedge \bigwedge_{j \neq i} \mathsf{fix}_j & \text{if } \alpha = i^{--}, \\ \Box_v^+ \neg p_i \wedge \bigwedge_{j < n} \mathsf{fix}_j & \text{if } \alpha = i^{??}. \end{cases} \tag{8.4}$$

Next, for each state $q \in Q$, we introduce a fresh propositional variables S_q, and take $\varphi_{\mathcal{M}}$ to be the conjunction of the following formula:

$$\widehat{S}_{q_{\mathsf{init}}} \;\wedge\; \bigwedge_{i<n} \Box_v^+ \neg p_i, \tag{8.5}$$

$$\Box_h^u \bigwedge_{q \in Q - H} \Big(\widehat{S}_q \;\rightarrow \bigvee_{(q,\alpha,q') \in \Delta} \big(\Box_h \widehat{S}_{q'} \wedge \mathsf{Do}_\alpha\big)\Big), \tag{8.6}$$

$$\Box_h^u \bigwedge_{h \in H} \neg \widehat{S}_h, \tag{8.7}$$

where $\widehat{S}_q := S_q \wedge \bigwedge_{q' \neq q} \neg S_{q'}$.

The first conjunct specifies the initial configuration $(q_{\mathsf{init}}, \vec{0}) \in \mathrm{Conf}_{\mathcal{M}}$ of $\mathcal{M}$, while the second governs the behaviour of the machine in accordance to the instructions of $\mathcal{M}$. The last conjunct stipulates that the computation be non-terminating. The following lemma makes these intuitions precise.

Lemma 8.3 (Emulation Lemma). *Suppose $\mathfrak{M}, (r_h, r_v) \models \varphi_{\mathcal{M}}$ and let $\langle x_k \in W_h : k < \omega\rangle$ be any infinite sequence such that $x_0 = r_h$ and $x_k R_h x_{k+1}$ for all $k < \omega$. Then $\mathcal{M}$ has a non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ such that $\mathfrak{M}, (x_k, r_v) \models \widehat{S}_{q_k}$, for all $k < \omega$.*

Proof. We construct, by induction on the length, an infinite sequence of configurations $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ such that $q_0 = q_{\mathsf{init}}$, $v_0 = \vec{0}$, and for all $k < \omega$:

(i) $\mathfrak{M}, (x_k, r_v) \models \widehat{S}_{q_k}$,

(ii) $v_k(i) = |\Sigma_i(x_k)|$, for all $i < n$,

(iii) If $k > 0$ then $(q_{k-1}, v_{k-1}) \xrightarrow{\mathcal{M}} (q_k, v_k)$.

First, by (8.5), we have that $\mathfrak{M}, (x_0, r_v) \models \widehat{S}_{q_{\mathsf{init}}}$ and $\mathfrak{M}, (x_0, r_v) \models \Box_v^+ \neg p_i$, for all $i < n$, where $x_0 = r_h$. Hence we may take, as our first configuration, the tuple (q_0, v_0), where $q_0 = q_{\mathsf{init}}$ and $v_0 = \vec{0}$.

Now suppose that $(q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}}$ has already been defined, for some $k < \omega$. By the induction hypothesis, we have that $\mathfrak{M}, (x_k, r_v) \models \widehat{S}_{q_k}$, while it follows from (8.19) that $q_k \notin H$. Thus, we may infer from (8.6) that

$$\mathfrak{M}, (x_k, r_v) \models \Box_h \widehat{S}_{q_{k+1}} \wedge \mathsf{Do}_\alpha,$$

for some $(q_k, \alpha, q_{k+1}) \in \Delta$. Hence we have that $\mathfrak{M}, (x_{k+1}, r_v) \models \widehat{S}_{q_{k+1}}$, since $x_k R_h x_{k+1}$, thereby satisfying (i).

We define $v_{k+1} : n \to \omega$ by taking

$$v_{k+1}(i) = |\Sigma_i(k+1)|,$$

for all $i < n$, thereby satisfying (ii).

It remains to show that $(q_k, v_k) \xrightarrow{\mathcal{M}} (q_{k+1}, v_{k+1})$. So suppose that $i < n$, and consider each of the four following cases, each of which follows from Lemma 8.2 and the induction hypothesis.

– If $\alpha = i^{++}$, then by (8.4) we have that $\mathfrak{M}, (x_k, r_v) \models \mathsf{inc}_i$. It then follows that

$$v_{k+1}(i) = |\Sigma_i(x_{k+1})| = |\Sigma_i(x_k)| + 1 = v_k(i) + 1.$$

– If $\alpha = i^{--}$, then by (8.4) we have that $\mathfrak{M}, (x_k, r_v) \models \mathsf{dec}_i$. It then follows that

$$v_{k+1}(i) = |\Sigma_i(x_{k+1})| = |\Sigma_i(x_k)| - 1 = v_k(i) - 1.$$

– If $\alpha = i^{??}$, then by (8.4) we have that $\mathfrak{M}, (x_k, r_v) \models \Box_v^+ \neg p_i \wedge \mathsf{fix}_i$. It then follows that

$$v_k(i) = |\Sigma_i(x_k)| = 0 \qquad \text{and} \qquad v_{k+1}(i) = |\Sigma_i(x_{k+1})| = |\Sigma_i(x_k)| = v_k(i).$$

– In all other cases where $\alpha \in \{j^{++}, j^{--}, j^{??}\}$ for $j \neq i$, we find that $\mathfrak{M}, (x_k, r_v) \models \mathsf{fix}_i$. It then follows that

$$v_{k+1}(i) = |\Sigma_i(x_{k+1})| = |\Sigma_i(x_k)| = v_k(i).$$

Thus, we conclude that $(q_k, v_k) \xrightarrow{\mathcal{M}} (q_{k+1}, v_{k+1})$, thereby satisfying (iii). Hence, by induction on the length of the sequence, we can construct an appropriate non-terminating computation for $\mathcal{M}$, as required. $\Box$

With this, we are now in a position to prove Theorem 8.1.

Proof of Theorem 8.1. Let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, and take

$$\psi_{\mathcal{M}} \;:=\; \Box_h^u \Diamond_h \top \wedge \varphi_{\mathcal{M}}.$$

We claim that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if $\mathcal{M}$ has a non-terminating computation.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable. Then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$, for some model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h, R_h') \in \mathcal{C}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

Since $\mathfrak{M}, (r_h, r_v) \models \Box_h^u \Diamond_h \top$, we can inductively generate an infinite ascending chain $\{x_k \in W_h : k < \omega\}$ such that $x_0 = r_h$ and $x_k R_h x_{k+1}$, for all $k < \omega$, as can be easily verified.

It then follows immediately from Lemma 8.3 that $\mathcal{M}$ has a non-terminating computation.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has a non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$. We define a model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (\omega, S, \omega^2) \in \mathcal{C}$ and

$\mathfrak{F}_v = (\omega, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, by taking

$$\begin{array}{rcll}\mathfrak{V}(S_q) & = & \{(k,0) : k < \omega \text{ and } q = q_k\}, & \text{for each } q \in Q,\\ \mathfrak{V}(p_i) & = & \{(k,m) : k < \omega \text{ and } m < v_k(i)\}, & \text{for each } i < n.\end{array}$$

It is then straightforward to check that $\mathfrak{M}, (0,0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable.

Now, since the Termination problem for counter machines is Σ^0_1-hard, so too must be the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$, as required. □

It now follows from Theorem 8.1 that $\mathbf{K}_u \times \mathbf{Diff}$ is undecidable; taking $\mathcal{C}$ to be the class of all frame for $\mathbf{K}_u$, which contains (ω, S, ω^2). Moreover, since both $\mathsf{Fr}\,\mathbf{K}_u$ and $\mathsf{Fr}\,\mathbf{Diff}$ are definable by means of a recursive (indeed, finite) set of first-order formulas, the recursive enumerability of $\mathbf{K}_u \times \mathbf{Diff}$ is a immediate corollary of Theorem 3.5.

Corollary 8.4. *The decision problem for $\mathbf{K}_u \times \mathbf{Diff}$ is Σ^0_1-complete.*

8.2.2 Transitive Closure Operator

It is tempting to consider the possibility of using the Reachability problem, in place of the non-Termination problem, in the proof of Theorem 8.1; constructing, instead, a formula that is satisfiable if and only if $\mathcal{M}$ has a computation reaching a given control state $\ell \in Q$. One might imagine that something akin to

$$\psi_{\mathcal{M}} \;:=\; \Box^u_h \Diamond_h \top \wedge \varphi_{\mathcal{M}} \wedge \Diamond^u_h \widehat{S}_\ell$$

would achieve this goal?

Of course, such a proposal would provide us with a non-recursively enumerable lower bound for $\mathbf{K}_u \times \mathbf{Diff}$, violating Theorem 3.5. The problem here, is that the interpretation of the universal modality may well extend far beyond the set of points reachable by any finite sequence of R_h-transitions.

However, in cases where we have not just a universal modality, but rather a *'master modality'* or *'common knowledge operator'*, interpreted by the reflexive-transitive closure of R_h [14], we obtain a much stronger result.

For a given Kripke complete unimodal logic L, let L_C denote the bimodal logic, having modal operators $\Diamond$ and $\Diamond^*$, characterised by all those frames of the form (W, R, R^*), where (W, R) is a frame for L and R^* denotes the reflexive-transitive closure of R. It is known that every such frame validates the following formulas, known as *Segerberg's axioms* [113]:

$$(seg_1) \quad := \quad \Box^* p \leftrightarrow (p \wedge \Box\Box^* p),$$

$$(seg_2) \quad := \quad \Box^*(p \to \Box p) \to (p \to \Box^* p),$$

which, together, corresponds to the stipulation that the relation interpreting $\Diamond^*$ should be precisely the reflexive-transitive closure of the relation interpreting $\Diamond$. It follows that if (W, R_1, R_2) is a frame for L_C then R_2 is the reflexive-transitive closure of R_1.

While it is the case that $\mathbf{K}_C \times \mathbf{S5}$ is decidable in CON2EXPTIME [38, Theorem 6.49], the following modification of Theorem 8.1 reveals that not only is $\mathbf{K}_C \times \mathbf{Diff}$ undecidable, but that its decision problem is not even analytic[†].

Theorem 8.5 (Hampson-Kurucz [58]). *Let $\mathcal{C}$ be any class of bimodal frames for $\mathbf{K}_C$ such that $(\omega, S, \leq) \in \mathcal{C}$, where S is the successor relation on ω. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Π^1_1-hard.*

Proof. We prove that the BÜCHI problem for counter machines is reducible to satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. To this end, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, with $\ell \in Q$, and define

$$\psi_{\mathcal{M}} \;:=\; \varphi_{\mathcal{M}} \wedge \Box_h^* \Diamond_h \Diamond_h^* \widehat{S}_\ell.$$

We claim that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a reliable computation of $\mathcal{M}$ in which ℓ occurs infinitely often.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$ for some model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h, R_h^*) \in \mathcal{C}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

Since $\mathfrak{M}, (r_h, r_v) \models \Box_h^* \Diamond_h \Diamond_h^* \widehat{S}_\ell$ we may inductively define an infinite sequence $\langle \hat{x}_k \in W_h : k < \omega \rangle$ such that $\hat{x}_0 = r_h$, and for all $k < \omega$:

- $\hat{x}_k R_h^* \hat{x}_{k+1}$ and $\hat{x} \neq \hat{x}_{k+1}$,

[†]This result improves upon the Π^0_1-hardness result given in [58], which employed a similar reduction from the REACHABILITY problem for reliable counter machines.

– $\mathfrak{M}, (\hat{x}_k, r_v) \models \widehat{S}_\ell$.

However, since R_h^* is the reflexive-transitive closure of R_h, we may extend this sequence to some sequence $\langle x_k \in W_h : k < \omega\rangle$ such that $x_0 = r_h$ and $x_k R_h x_{k+1}$, where $\mathfrak{M}, (x_k, r_v) \models \widehat{S}_\ell$, for infinitely many $k < \omega$. That is to say that, for every $\hat{x}_k R_h^* \hat{x}_{k+1}$, we insert finitely many x_j's connecting $\hat{x}_k$ to $\hat{x}_{k+1}$.

By Lemma 8.3, there is a some computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ of $\mathcal{M}$ such that,

$$\mathfrak{M}, (x, r_v) \models \widehat{S}_{q_k},$$

for all $k < \omega$.

However, since $\mathfrak{M}, (x_k, r_v) \models \widehat{S}_\ell$, for infinitely many $k < \omega$, it follows that there is a computation of $\mathcal{M}$ in which ℓ occurs infinitely often.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has a computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ in which ℓ occurs infinitely often. Take $\mathfrak{M}$ to be the model defined in the proof of Theorem 8.1. It is then straightforward to check that $\mathfrak{M}, (0, 0) \models \psi_{\mathcal{M}}$.

Now, since the BÜCHI problem for counter machines is Σ^1_1-hard, so too must be the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. Hence, the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ must be Π^1_1-hard, as required. $\Box$

It now follows from Theorem 8.5 that the decision problem for $\mathbf{K}_C \times \mathbf{Diff}$ is Π^1_1-hard; taking $\mathcal{C}$ to be the class of all frames for $\mathbf{K}_C$, which contains $(\omega, S, \leq)$. Thus we have the following corollary.

Corollary 8.6. *The decision problem for* $\mathbf{K}_C \times \mathbf{Diff}$ *is* Π^1_1*-hard.*

Propositional temporal logic $\mathbf{PTL}_{\bigcirc\Box}$, with 'future' and 'next-time' operators is the logic characterised by the frame $(\omega, S, \leq)$. It is proved in [38, Theorem 6.29] that $\mathbf{PTL}_{\bigcirc\Box} \times L$ is characterised by the class of frames $(\omega, S, \leq) \times \mathsf{Fr}\, L$, whenever L is any Kripke complete modal logic whose frames are first-order definable in the language having equality and a binary predicate symbol. Hence, we obtain, as a further corollary of Theorem 8.5, that the decision problem for $\mathbf{PTL}_{\bigcirc\Box} \times \mathbf{Diff}$, too, is non-analytic.

Corollary 8.7. *The decision problem for* $\mathbf{PTL}_{\bigcirc\Box} \times \mathbf{Diff}$ *is* Π^1_1*-hard.*

This is perhaps a more striking result since the decision problem for $\mathbf{PTL}_{\bigcirc\Box} \times \mathbf{S5}$ is known to be EXPSPACE-complete [38, Theorem 6.65].

8.3 Products without 'Next-time'

In the previous section, we considered only those cases in which we were afforded both horizontal 'universal' and 'next-time' operators. It is tempting to think that this places fairly restrictive limitations on this technique — especially given the decidability of $\mathbf{K} \times \mathbf{Diff}$, discussed in Section 6.3, which has only a single horizontal modality at its disposal.

In this section, we show that this technique can also be effective in cases where the horizontal dimension is characterised by some class of (unimodal) linear orders *without* a 'next-time' operator. However, it is possible to overcome this deficiency by constructing an ascending sequence of 'diagonal' points, with each 'step' along the diagonal being used to emulate the transition of control states of our chosen machine.

Unfortunately, this limited use of a 'next-time' operator does not extend to our previous emulation of our 'counting mechanism' whereby a single propositional variable was reserved for each counter. Instead we must construct a new counting mechanism that does not require the use of a 'next-time' operator.

8.3.1 Linear Orders

A frame $\mathfrak{F} = (W, R)$ is said to be a *linear order* if it satisfies the following first-order conditions:

- *transitivity:* $\forall x \forall y \forall z \; \big(xRy \wedge yRz \rightarrow xRz\big)$,
- *antisymmetry:* $\forall x \forall y \; \big(xRy \wedge yRx \rightarrow x = y\big)$,
- *connectivity:* $\forall x \forall y \; \big(xRy \vee x = y \vee yRx\big)$.

A linear order is said to be *strict* if there is no $x \in W$ such that xRx, and *dense* if for all $x, y \in W$ there is some $z \in W$ such that xRz and zRy. A *well-order* is any linear order for which every non-empty subset of W contains an element that is minimal with respect to R. For all $x, y \in W$, we say that y is the *immediate R-successor* of x if xRy and there is no $z \in W$ such that xRz and zRy. We define the *immediate R-predecessor* of x, analogously. Call a sequence $\langle x_k \in W : k < L\rangle$, of length $L \leq \omega$, a *strictly ascending R-chain* if $x_{k-1} R x_k$ and $x_{k-1} \neq x_k$, for all $0 < k < L$.

The following result answers a long standing question posed by Reynolds [101] and others [83, 102], as well as generalising many pre-existing results relating to products in which one component is characterised by some class of linear orders; to be discussed in Section 8.4.

Theorem 8.8 (Hampson-Kurucz [59]). *Let $\mathcal{C}$ be any class of strict linear orders such that $(\omega, <) \in \mathcal{C}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Σ^0_1-hard.*

As above, we fix an arbitrary reliable counter machine $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$, and let $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$ be a rooted product model, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ is a strict linear order and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

The first task we face with transitive frames is the lack of a meaningful 'next time' operation. For while we may easily specify the global behaviour of our counter machine, we cannot so easily speak about the *immediate* R_h-successor of a given state without some additional structure.

To this end, let c, d be propositional variables and take grid^{fw} to be the following formula:

$$c \;\wedge\; \Box_h^+ \Diamond_v (d \wedge \Diamond_h c \wedge \neg \Diamond_h \Diamond_h c). \tag{8.8}$$

This formula imposes upon $\mathfrak{M}$ the existence of an infinite grid, as illustrated in Figure 8.1.

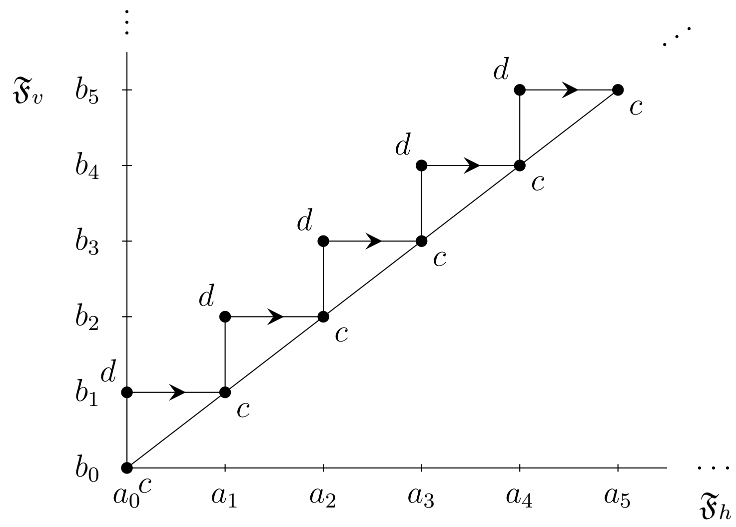


Figure 8.1: Illustration of the grid generated by grid^{fw}.

Lemma 8.9. *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw}$. *Then there are two infinite sequences* $\langle a_k \in W_h : k < \omega\rangle$ *and* $\langle b_k \in W_v : k < \omega\rangle$, *such that, for all* $k < \omega$*:*

(i) $a_0 = r_h$ *and if* $k > 0$ *then* a_k *is the immediate* R_h*-successor of* a_{k-1},

(ii) $b_0 = r_v$ *and if* $k > 0$ *then* $r_v R_v b_k$,

(iii) $\mathfrak{M}, (a_k, b_k) \models c$,

(iv) *If* $k > 0$ *then* $\mathfrak{M}, (a_{k-1}, b_k) \models d$.

Proof. First let $a_0 = r_h$ and $b_0 = r_v$, so that $\mathfrak{M}, (a_0, b_0) \models c$, as required.

Now suppose that we have already defined $a_k \in W_h$, $b_k \in W_v$, for some $k < \omega$. By (i), we have that $r_h R_h^+ a_k$ and so it follows from (8.8) that there is some $b_{k+1} \in W_v$ such that $r_v R_v b_{k+1}$ and $\mathfrak{M}, (a_k, b_{k+1}) \models d \wedge \Diamond_h c \wedge \neg\Diamond_h\Diamond_h c$. Hence, there must be some $a_{k+1} \in W_h$ such that $a_k R_h a_{k+1}$ and $\mathfrak{M}, (a_{k+1}, b_{k+1}) \models c$. Moreover, a_{k+1} must be the immediate R_h-successor of a_k since $\mathfrak{M}, (a_k, b_{k+1}) \models \neg\Diamond_h\Diamond_h c$. Hence, by induction on the length, we can construct two appropriate sequences, as required. $\square$

Rather than reserving a single variable to represent each counter, we introduce *two* counter variables p_i and q_i, for each $i < n$. The intuition behind this is that p_i marks the instances where the counter i is incremented, and remains true thereafter; subsequent decrements to the counter i are marked by 'covering' p_i with the variable q_i. Hence the true value of the counter will be represented by those places marked by $(p_i \wedge \neg q_i)$.

To be more precise, for each $i < n$, we define

$$\Sigma_i(x) = \{y \in W_v : r_v R_v^+ y \text{ and } \mathfrak{M}, (x, y) \models p_i \wedge \neg q_i\}, \tag{8.9}$$

for all $x \in W_h$. The value of counter i, held at instance $x \in W_h$, is then captured by the cardinality of $\Sigma_i(x)$.

To facilitate these changes we also introduce the variables $\mathsf{start}(p_i)$ and $\mathsf{start}(q_i)$ to mark those instances where we first satisfy p_i and q_i, respectively[†]. For each $p \in \{p_i, q_i : i < n\}$

[†]Strictly speaking, these additional variables are redundant, as can be seen from the proof given in [59]. However their inclusion here aids the simplicity of the subsequent proofs, and will be invaluable in Section 8.3.4

take $\mathsf{counter}(p)$ to be the conjunction of the following formulas:

$$\Box_h^+\Box_v^+\big(\mathsf{start}(p) \leftrightarrow (\neg p \wedge \Box_h p)\big), \tag{8.10}$$

$$\Box_h^+\Box_v^+(p \to \Box_h p). \tag{8.11}$$

The effect of which is illustrated by the following lemma.

Lemma 8.10. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{counter}(p)$ and let $x, x' \in W_h$ and $y \in W_v$ be such that $r_h R_h^+ x$, $r_v R_v^+ y$ and x' is the immediate R_h-successor of x. Then we have that:*

(i) *If $\mathfrak{M}, (x, y) \models \mathsf{start}(p)$ then $\mathfrak{M}, (x, y) \models \neg p$ and $\mathfrak{M}, (x', y) \models p$,*

(ii) *If $\mathfrak{M}, (x, y) \models \neg\mathsf{start}(p)$ then $\mathfrak{M}, (x, y) \models p$ if and only if $\mathfrak{M}, (x', y) \models p$.*

Proof. (i) Suppose that $\mathfrak{M}, (x, y) \models \mathsf{start}(p)$. It then follows from (8.10) that $\mathfrak{M}, (x, y) \models \neg p \wedge \Box_h p$. Hence, we have that $\mathfrak{M}, (x', y) \models p$, since $x R_h x'$, as required.

(ii) Suppose that $\mathfrak{M}, (x, y) \models \neg\mathsf{start}(p)$. By (8.10), we have that $\mathfrak{M}, (x, y) \models \Box_h p \to p$. Suppose that $\mathfrak{M}, (x', y) \models p$. Then by (8.11), we have that $\mathfrak{M}, (x, y) \models \Box_h p$, since x' is the immediate R_h-successor of x. It then follows that $\mathfrak{M}, (x, y) \models p$. Conversely, suppose that $\mathfrak{M}, (x, y) \models p$. Then it is immediate from (8.11) that $\mathfrak{M}, (x', y) \models p$, since $x R_h x'$, as required.

□

Take $\mathsf{counter}$ to be the conjunction of $\mathsf{counter}(p)$, for all $p \in \{p_i, q_i : i < n\}$, together with the following formula, stipulating that we cannot mark a counter as 'off' before we first mark it as 'on':

$$\bigwedge_{i<n} \Box_h^+\Box_v^+(q_i \to p_i). \tag{8.12}$$

With these prerequisite definitions, we may easily define the behaviour of each of the possible counter operations of Op_n. For each $i < n$ we define the following formulas:

$$\mathsf{fix}_i^{fw} \quad := \quad \Box_v^+\neg\mathsf{start}(p_i) \wedge \Box_v^+\neg\mathsf{start}(q_i), \tag{8.13}$$

$$\mathsf{inc}_i^{fw} \quad := \quad \Diamond_v^{=1}\mathsf{start}(p_i) \wedge \Box_v^+\neg\mathsf{start}(q_i), \tag{8.14}$$

$$\mathsf{dec}_i^{fw} \quad := \quad \Diamond_v^{=1}\mathsf{start}(q_i) \wedge \Box_v^+\neg\mathsf{start}(p_i). \tag{8.15}$$

The interpretation of which is explained by the following analogue of Lemma 8.2.

Lemma 8.11 (Counting Lemma)**.** *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw} \wedge \mathsf{counter}$*. Then for all* $k < \omega$*:*

(i) *If* $\mathfrak{M}, (a_k, r_v) \models \mathsf{fix}_i^{fw}$*, then* $\Sigma_i(a_{k+1}) = \Sigma_i(a_k)$*,*

(ii) *If* $\mathfrak{M}, (a_k, r_v) \models \mathsf{inc}_i^{fw}$*, then* $\Sigma_i(a_{k+1}) = \Sigma_i(a_k) \cup \{z\}$ *for some* $z \notin \Sigma_i(a_k)$*,*

(iii) *If* $\mathfrak{M}, (a_k, r_v) \models \mathsf{dec}_i^{fw}$*, then* $\Sigma_i(a_{k+1}) = \Sigma_i(a_k) - \{z\}$ *for some* $z \in \Sigma_i(a_k)$*.*

Proof. (i) Suppose that $y \in \Sigma_i(a_{k+1})$. Then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$. By (8.13), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(p_i) \wedge \neg\mathsf{start}(q_i)$. It then follows from Lemma 8.10 that $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$, since a_{k+1} is the immediate R_h-successor of a_k. This is to say that $y \in \Sigma_i(a_k)$.

Conversely, suppose that $y \in \Sigma_i(a_k)$. Then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$. By (8.13), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(p_i) \wedge \neg\mathsf{start}(q_i)$. Again, it then follows from Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$, since a_{k+1} is the immediate R_h-successor of a_k. This is to say that $y \in \Sigma_i(a_{k+1})$.

(ii) By (8.14), there is some unique $z \in W_v$ such that $r_v R_v^+ z$ and $\mathfrak{M}, (a_k, z) \models \mathsf{start}(p_i) \wedge \neg\mathsf{start}(q_i)$. By Lemma 8.10 we have that $\mathfrak{M}, (a_k, z) \models \neg p_i$ and $\mathfrak{M}, (a_{k+1}, z) \models p_i$, since a_{k+1} is the immediate R_h-successor of a_k. It follows from (8.12) that $\mathfrak{M}, (a_k, z) \models \neg q_i$ and so, by Lemma 8.10, $\mathfrak{M}, (a_{k+1}, z) \models \neg q_i$. Hence $z \notin \Sigma_i(a_k)$ and $z \in \Sigma_i(a_{k+1})$.

Now suppose that $y \in \Sigma_i(a_{k+1})$. Then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$. If $y \neq z$ then it follows from (8.14) that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(p_i) \wedge \neg\mathsf{start}(q_i)$.

It then follows from Lemma 8.10 that $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$, since a_{k+1} is the immediate R_h-successor of a_k. This is to say that $y \in \Sigma_i(a_k)$.

Conversely, suppose that $y \in \Sigma_i(a_k)$. Then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$.

We know that $y \neq z$ since $z \notin \Sigma_i(a_k)$, and so it follows from (8.14) that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(p_i) \wedge \neg\mathsf{start}(q_i)$. Again, it then follows from Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$, since a_{k+1} is the immediate R_h-successor of a_k. This is to say that $y \in \Sigma_i(a_{k+1})$.

(iii) By (8.15), there is some unique $z \in W_v$ such that $r_v R_v^+ z$ and $\mathfrak{M}, (a_k, z) \models \neg\mathsf{start}(p_i) \wedge \mathsf{start}(q_i)$. By Lemma 8.10 we have that $\mathfrak{M}, (a_k, z) \models \neg q_i$ and $\mathfrak{M}, (a_{k+1}, z) \models q_i$, since a_{k+1} is the immediate R_h-successor of a_k. It follows from (8.12) that $\mathfrak{M}, (a_{k+1}, z) \models p_i$ and so, by Lemma 8.10, $\mathfrak{M}, (a_k, z) \models p_i$. Hence $z \in \Sigma_i(a_k)$ and $z \notin \Sigma_i(a_{k+1})$.

Now suppose that $y \in \Sigma_i(a_{k+1})$. Then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$. We know that $y \neq z$ since $z \notin \Sigma_i(a_{k+1})$, and so it follows from (8.15) that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(p_i) \wedge \neg\mathsf{start}(q_i)$. It then follows from Lemma 8.10 that $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$, since a_{k+1} is the immediate R_h-successor of a_k. This is to say that $y \in \Sigma_i(a_k)$.

Conversely, suppose that $y \in \Sigma_i(a_k)$ and $y \neq z$. Then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$. By (8.15), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(p_i) \wedge \neg\mathsf{start}(q_i)$, since $y \neq z$. Again, it then follows from Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$, since a_{k+1} is the immediate R_h-successor of a_k. This is to say that $y \in \Sigma_i(a_{k+1})$.

□

We then specify the result of each of the counter operation $\alpha \in Op_n$ as follows:

$$\mathsf{Do}_\alpha^{fw} := \begin{cases} \mathsf{inc}_i^{fw} \wedge \bigwedge_{j \neq i} \mathsf{fix}_j^{fw} & \text{if } \alpha = i^{++}, \\ \mathsf{dec}_i^{fw} \wedge \bigwedge_{j \neq i} \mathsf{fix}_j^{fw} & \text{if } \alpha = i^{--}, \\ \Box_v^+(p_i \to q_i) \wedge \bigwedge_{j<n} \mathsf{fix}_j^{fw} & \text{if } \alpha = i^{??}. \end{cases} \tag{8.16}$$

As above, we introduce a fresh propositional variable S_q, for each control state $q \in Q$, and take $\varphi_{\mathcal{M}}$ to be the conjunction of the following formulas:

$$\widehat{S}_{q_{\mathsf{init}}} \wedge \bigwedge_{i<n} \Box_v^+ \neg p_i, \tag{8.17}$$

$$\Box_h^+ \bigwedge_{q \in Q-H} \Big(\Diamond_v^+ (c \wedge \widehat{S}_q) \to \bigvee_{(q,\alpha,q') \in \Delta} \big(\mathsf{Do}_\alpha^{fw} \wedge \Box_v^+ \big(d \to \Box_h (c \to \widehat{S}_{q'}) \big) \big) \Big), \tag{8.18}$$

$$\Box_h^+ \Box_v^+ \bigwedge_{h \in H} \neg \widehat{S}_h, \tag{8.19}$$

where $\widehat{S}_q := S_q \wedge \bigwedge_{q' \neq q} \neg S_{q'}$.

The first conjunct specifies the initial configuration $(q_{\mathsf{init}}, \vec{0}) \in \mathrm{Conf}_{\mathcal{M}}$, while the second governs the behaviour of the machine in accordance to the instructions of $\mathcal{M}$. The last conjunct stipulates that the computation be non-terminating. These details are more formally addressed by the following lemma.

Lemma 8.12 (Emulation Lemma). *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}}$ *and let* $\langle a_k \in W_h : k < \omega\rangle$ *and* $\langle b_k \in W_v : k < \omega\rangle$ *be any infinite sequences satisfying conditions (i)–(iv) of Lemma 8.9. Then* $\mathcal{M}$ *has a non-terminating computation* $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ *such that* $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$*, for all* $k < \omega$.

Proof. We construct, by induction on the length, an infinite sequence of configuration $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ such that $q_0 = q_{\mathsf{init}}$, $v_0 = \vec{0}$, and for all $k < \omega$:

(i) $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$,

(ii) $v_k(i) = |\Sigma_i(a_k)|$, for all $i < n$,

(iii) If $k > 0$ then $(q_{k-1}, v_{k-1}) \xrightarrow{\mathcal{M}} (q_k, v_k)$.

Firstly, by Lemma 8.9 we have that $r_h = a_0$, $r_v = b_0$, while by (8.5), we have that $\mathfrak{M}, (a_0, b_0) \models \widehat{S}_{q_{\mathsf{init}}}$ and $\mathfrak{M}, (a_0, b_0) \models \Box_v^+ \neg p_i$, for all $i < n$. Hence we may take, as our first configuration, the tuple (q_0, v_0), where $q_0 = q_{\mathsf{init}}$ and $v_0 = \vec{0}$.

Now suppose that $(q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}}$ has already been defined, for some $k < \omega$. By the induction hypothesis, we have that $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$, while it follows from (8.19) that $q_k \notin H$. Furthermore, by Lemma 8.9 we have that $\mathfrak{M}, (a_k, r_h) \models \Diamond_v (c \wedge \widehat{S}_{q_k})$.
Thus, we may infer from (8.18) that

$$\mathfrak{M}, (a_k, r_v) \models \mathsf{Do}_{\alpha}^{fw} \wedge \Box_v^+ \big(d \to \Box_h (c \to \widehat{S}_{q_{k+1}})\big),$$

for some $(q, \alpha, q_{k+1}) \in \Delta$. It then follows from Lemma 8.9 that $\mathfrak{M}, (a_{k+1}, b_{k+1}) \models \widehat{S}_{q_{k+1}}$, thereby satisfying (i).

We define $v_{k+1} : n \to \omega$ by taking

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})|,$$

for all $i < n$, thereby satisfying (ii).

It remains to show that $(q_k, v_k) \xrightarrow{\mathcal{M}} (q_{k+1}, v_{k+1})$. Suppose that $i < n$, and consider each of the four following cases, each of which follows from Lemma 8.11 and the induction hypothesis:

– If $\alpha = i^{++}$, then by (8.16) we have that $\mathfrak{M}, (a_k, r_v) \models \mathsf{inc}_i^{fw}$. It then follows that

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})| = |\Sigma_i(a_k)| + 1 = v_k(i) + 1.$$

– If $\alpha = i^{--}$, then by (8.16) we have that $\mathfrak{M}, (a_k, r_v) \models \mathsf{dec}_i^{fw}$. It then follows that

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})| = |\Sigma_i(a_k)| - 1 = v_k(i) - 1.$$

– If $\alpha = i^{??}$, then by (8.16) we have that $\mathfrak{M}, (a_k, r_v) \models \Box_v^+(p_i \to q_i) \wedge \mathsf{fix}_i^{fw}$. It then follows that

$$v_k(i) = |\Sigma_i(a_k)| = 0 \qquad \text{and} \qquad v_{k+1}(i) = |\Sigma_i(a_{k+1})| = |\Sigma_i(a_k)| = v_k(i).$$

– In all other cases where $\alpha \in \{j^{++}, j^{--}, j^{??}\}$ for $j \neq i$, we find that $\mathfrak{M}, (a_k, r_v) \models \mathsf{fix}_i^{fw}$. It then follows that

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})| = |\Sigma_i(a_k)| = v_k(i).$$

Thus, we conclude that $(q_k, v_k) \xrightarrow{\mathcal{M}} (q_{k+1}, v_{k+1})$, thereby satisfying (iii). Hence, by induction on the length of the sequence, we can construct an appropriate non-terminating computation for $\mathcal{M}$, as required. $\square$

We are now in a position to prove Theorem 8.8.

Proof of Theorem 8.8. We prove that the non-Termination problem for reliable counter machines is reducible to the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. To this end, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, and define

$$\psi_{\mathcal{M}} \;:=\; \mathsf{grid}^{fw} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}}$$

It remains to show that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a non-terminating computation of $\mathcal{M}$.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$ for some model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

It follows immediately from Lemmas 8.11 and 8.12, that $\mathcal{M}$ has a non-terminating computation.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has a non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$. We define a product model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (\omega, <) \in \mathcal{C}$ and $\mathfrak{F}_v = (\omega, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, by taking

$$\begin{aligned}
\mathfrak{V}(c) &= \{(k,k) : k < \omega\},\\
\mathfrak{V}(d) &= \{(k,k-1) : 0 < k < \omega\},\\
\mathfrak{V}(S_q) &= \{(k,k) : k < \omega \text{ and } q_k = q\}, \quad \text{for each } q \in Q.
\end{aligned}$$

We define, for each $i < n$, the functions $\eta_i^+, \eta_i^- : \omega \to \omega$, by taking $\eta_i^+(0) = \eta_i^-(0) = 0$ and, for all $k < \omega$,

$$\begin{aligned}
\eta_i^+(k+1) &= \begin{cases} \eta_i^+(k) + \big(v_{k+1}(i) - v_k(i)\big) & \text{if } v_{k+1}(i) > v_k(i),\\ \eta_i^+(k) & \text{otherwise,}\end{cases}\\
\eta_i^-(k+1) &= \begin{cases} \eta_i^-(k) + \big(v_k(i) - v_{k+1}(i)\big) & \text{if } v_{k+1}(i) < v_k(i),\\ \eta_i^-(k) & \text{otherwise.}\end{cases}
\end{aligned}$$

This is to say that η_i^+ records the cumulative sum of all the increments made to counter i, while η_i^- records the cumulative sum of all the decrements. It follows by a simple induction that $\big(\eta_i^+(k) - \eta_i^-(k)\big) = v_k(i)$, for all $k < \omega$.

We then define the valuations of p_i and q_i, by taking

$$\begin{aligned}
\mathfrak{V}(p_i) &= \{(k,m) : k < \omega \text{ and } m < \eta_i^+(k)\},\\
\mathfrak{V}(q_i) &= \{(k,m) : k < \omega \text{ and } m < \eta_i^-(k)\},
\end{aligned}$$

for all $i < n$. It follows that $m \in \Sigma_i(k)$ if and only if $(k,m) \in \mathfrak{V}(p_i) - \mathfrak{V}(q_i)$, which is to say that $\eta_i^-(k) \leq m < \eta_i^+(k)$. Therefore $|\Sigma_i(k)| = \eta_i^+(k) - \eta_i^-(k) = v_k(i)$.

Lastly we evaluate $\mathsf{start}(p)$ by taking, for all $k, m < \omega$,

$$\mathfrak{V}(\mathsf{start}(p)) = \big\{(k,m) : (k,m) \notin \mathfrak{V}(p) \text{ and } (k+1,m) \in \mathfrak{V}(p)\big\},$$

for each $p \in \{p_i, q_i : i < n\}$.

It is then straightforward to check that $\mathfrak{M}, (0,0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable.

Now, since the TERMINATION problem for reliable counter machines is Σ^0_1-hard, so too must be the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$, as required. $\Box$

By a straightforward bulldozing argument (see, for example [14]), every frame for **K4.3** is the p-morphic image of some strict linear order. Hence, it follows, as an immediate corollary of Theorem 8.8 and Proposition 2.4, that the decision problem for $\mathbf{K4.3} \times \mathbf{Diff}$ is undecidable; taking $\mathcal{C}$ to be the class of all strict linear orders. Moreover, by Theorem 3.5, we have that $\mathbf{K4.3} \times \mathbf{Diff}$ is recursively enumerable, since both $\mathsf{Fr}\,\mathbf{K4.3}$ and $\mathsf{Fr}\,\mathbf{Diff}$ are definable by some recursive set of first-order conditions.

Corollary 8.13. *The decision problem for* $\mathbf{K4.3} \times \mathbf{Diff}$ *is* Σ^0_1*-complete.*

Here, again, we see a great disparity between **S5**-products and **Diff**-products, comparing the modest 2EXPTIME complexity of the decision problem for $\mathbf{K4.3} \times \mathbf{S5}$ [101] with the undecidability of $\mathbf{K4.3} \times \mathbf{Diff}$.

8.3.2 Modally Discrete Linear Orders

A linear order $\mathfrak{F} = (W, R)$ is said to be *modally discrete* if, between any two points, there is no infinite *strictly ascending* R-chain. It is a straightforward exercise to show that an arbitrary *strict* linear order is modally discrete if and only if it validates the following axiom [111, 46]:

$$(dis) \;:=\; \Box(\Box p \to p) \to (\Diamond\Box p \to \Box p),$$

while it was proved by Prior [98] that an arbitrary *reflexive* linear order is modally discrete if and only if it validates *Dummett's axiom* (originating in [30]):

$$(dum) \;:=\; \Box\big(\Box(p \to \Box p) \to p\big) \to (\Diamond\Box p \to p).$$

We define **DisK4.3** to be the smallest normal extension of **K4.3** containing (dis), whose frames comprise the class of all modally discrete strict linear orders, while **S4.3Dum** is defined to be the smallest normal extension of **S4.3** containing (dum), whose frames comprise the class of all modally discrete *reflexive* linear orders.

The class of all modally discrete strict linear orders is not first-order definable, and as such, the complexity of the decision problems for both **DisK4.3** $\times$ **Diff** and **S4.3Dum** $\times$ **Diff** are not bound by the impositions of Theorem 3.5. Here we show that, indeed, for any class of modally discrete strict linear orders containing $(\omega, <)$, the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is not only undecidable, but non-analytic.

Treatment of reflexive linear orders will be postponed until Section 8.3.4, where we show that the decision problem for **S4.3Dum** $\times$ **Diff** is also non-analytic.

Theorem 8.14 (Hampson-Kurucz [59]). *Let $\mathcal{C}$ be any class of modally discrete strict linear orders such that $(\omega, <) \in \mathcal{C}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Π^1_1-hard.*

Proof. We prove that the Büchi problem for counter machines is reducible to the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. To this end, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, with $\ell \in Q$, and define

$$\psi_{\mathcal{M}} \;:=\; \mathsf{grid}^{fw} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}} \wedge \Box_h^+ \Diamond_h \Box_v^+ (c \to \widehat{S}_\ell).$$

We claim that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a computation of $\mathcal{M}$ in which ℓ occurs infinitely often.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable. Then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$, for some model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

By Lemma 8.12, there is a some non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$ of $\mathcal{M}$ such that, for all $k < \omega$,

$$\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k},$$

where $a_k \in W_h$ and $b_k \in W_v$ are as defined in Lemma 8.9.

Now, for each $m < \omega$, we have that $r_h R_h^+ a_m$ and so $\mathfrak{M}, (a_m, r_v) \models \Diamond_h \Box_v^+ (c \to \widehat{S}_\ell)$. Therefore, there is some $a \in W_h$ such that $a_m R_h a$ and $\mathfrak{M}, (a, r_v) \models \Box_v^+ (c \to \widehat{S}_\ell)$. Furthermore, since $\mathfrak{F}_h$ is modally discrete, we have that $a = a_j$ for some $m < j < \omega$, for otherwise the sequence $\langle a_k : m < k < \omega \rangle$ would form an infinite ascending chain between a_m and a. It then follows from Lemma 8.9 that $\mathfrak{M}, (a_j, r_v) \models \widehat{S}_\ell$. Thus we may conclude that there is some non-terminating computation of $\mathcal{M}$ in which ℓ occurs infinitely often.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has a non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$ in which ℓ occurs infinitely often. Take $\mathfrak{M}$ to be the model defined in the proof of Theorem 8.8. It is then straightforward to check that $\mathfrak{M}, (0,0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable.

Now, since the Büchi problem for counter machines is Σ^1_1-hard, so too must be the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. Hence, the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ must be Π^1_1-hard, as required. □

It follows immediately from Theorem 8.14 that $\mathbf{DisK4.3} \times \mathbf{Diff}$ is non-analytic; taking $\mathcal{C}$ to be the class of all modally discrete strict linear orders.

Corollary 8.15. *The decision problem for* $\mathbf{DisK4.3} \times \mathbf{Diff}$ *is* Π^1_1*-hard.*

As two further corollaries, we have that the decision problems for both $\mathsf{Log}(\omega, <) \times \mathbf{Diff}$ and $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{Diff})$ are Π^1_1-hard. The former being characterised by those product frames whose first component is a right-unbounded, modally discrete linear order and whose second component is a frame for $\mathbf{Diff}$. Note further that these two logics are distinct; despite their *prima facie* similarities. In [38, Theorem 6.29] it is wrongly stated that $\mathsf{Log}(\omega, <) \times \mathbf{S5}$ and $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{S5})$ are identical. However, this is shown to be incorrect in [72]; the same formula that distinguishes these two logics also distinguishes $\mathsf{Log}(\omega, <) \times \mathbf{Diff}$ from $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{Diff})$.

Corollary 8.16. *The decision problems for both* $\mathsf{Log}(\omega, <) \times \mathbf{Diff}$ *and* $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{Diff})$ *are* Π^1_1*-hard.*

Here, the non-analyticity of $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{Diff})$ stands in marked contrast to the ExpSpace-completeness of the decision problem for $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{S5})$ [38, Theorem 6.65]†. It is suspected that the decision problem for $\mathsf{Log}(\omega, <) \times \mathbf{S5}$ too is decidable; however, a correct proof is yet to appear.

Question 8.17. Is the decision problem for $\mathsf{Log}(\omega, <) \times \mathbf{S5}$ decidable?

†Here, this theorem is misstated as proving the ExpSpace-completeness of $\mathsf{Log}(\omega, <) \times \mathbf{S5}$, where, in truth, what is proved is that $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{S5})$ is ExpSpace-completeness.

These results highlight the discrepancy between the computational complexity of products and that of their constituent parts. It is well-known that the decision problem for both $\mathbf{K4.3}$ and $\mathsf{Log}(\omega, <)$ are coNP-complete [91] while the complexity of $\mathbf{K}_u$ is ExpSpace-complete [121], necessitating the existence of some polynomial reduction from $\mathsf{Log}(\omega, <)$ to $\mathbf{K}_u$. However, there can be no such reduction between their product logics, since the decision problem for $\mathsf{Log}(\omega, <) \times \mathbf{Diff}$ is overwhelmingly more complex than the more modest $\mathbf{K}_u \times \mathbf{Diff}$, whose decision problem is recursively enumerable.

8.3.3 Finite Linear Orders

The following standard result of [38], paraphrased here, provides us with a finite bound on the size of the product models for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$, whenever $\mathcal{C}$ comprises some class of finite frames.

Proposition 8.18. *Let L be any Kripke complete unimodal logic characterised by its finite frames and let $\mathcal{C}$ be any class of finite frames. Then $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,L)$ has the finite product model property.*

Proof. See Proposition 5.35 of [38]. □

It is well-known that $\mathbf{Diff}$ enjoys the polysize model property and is thus characterise by its finite frames [26]. Hence, it follows from Proposition 8.18 that $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ enjoys the product fmp, whenever $\mathcal{C}$ comprises a class of finite frames.

Indeed, should $\mathcal{C}$ also be recursive, then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ would be co-recursively enumerable; to check whether an arbitrary formula $\varphi \in \mathcal{ML}_2$ is valid in $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ it is enough to enumerate all finite models, based on product frames for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ — enumerable, since both $\mathcal{C}$ and $\mathsf{Fr}\,\mathbf{Diff}$ are recursive — and sequentially search for a refuting model.

For this reason, there can be no reduction from the Termination problem to the decision problem for such logics $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. For such a reduction would directly contradict their supposed co-recursive enumerability. However, over finite linear orders we may, instead, encode instances of the non-Reachability problem for a given counter machine, thereby providing us with the optimal Π^0_1-complete lower bound.

Theorem 8.19 (Hampson-Kurucz [59]). *Let $\mathcal{C}$ be the class of all finite strict linear orders. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Π^0_1-hard.*

First, we note that grid^{fw} is satisfiable only in those frames having an infinite ascending R_h-chain, in accordance with Lemma 8.9. However, the following variation of grid^{fw} provides the same service, while enjoying the benefit of finite satisfiability. We take grid^{fin} to be the formula:

$$c \wedge \Box_h^+ \big(\Diamond_h \top \to \Diamond_v (d \wedge \Diamond_h c \wedge \neg \Diamond_h \Diamond_h c)\big), \tag{8.20}$$

for which we have the following analogue of Lemma 8.9.

Lemma 8.20. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fin}$. Then there is some $L \leq \omega$ and two sequences $\langle a_k \in W_h : k < L\rangle$ and $\langle b_k \in W_v : k < L\rangle$ of length L, such that, for all $k < \omega$:*

(i) *$a_0 = r_h$ and if $k > 0$ then a_k is the immediate R_h-successor of a_{k-1},*

(ii) *$b_0 = r_v$ and if $k > 0$ then $r_v R_v b_k$,*

(iii) *$\mathfrak{M}, (a_k, b_k) \models c$,*

(iv) *If $k > 0$ then $\mathfrak{M}, (a_{k-1}, b_k) \models d$,*

(v) *$\mathfrak{M}, (a_k, r_v) \models \Box_h \bot$ if and only if $k + 1 = L$.*

Proof. The proof is analogous to that of Lemma 8.9. $\square$

For each $q \in Q$, let S_q be a fresh propositional variable and take $\varphi_{\mathcal{M}}$ to be the conjunction of the formulas given in (8.17)–(8.19), as defined in the proof of Theorem 8.8. The following emulation lemma is completely analogous to that of Lemma 8.12.

Lemma 8.21. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fin} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}}$, and let $\langle a_k \in W_h : k < L\rangle$ and $\langle b_k \in W_v : k < L\rangle$ be any sequence satisfying conditions (i)–(v) of Lemma 8.20, for some $L \leq \omega$. Then $\mathcal{M}$ has a computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L'\rangle$ of length $L' > L$, such that $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$, for all $k < L$.*

Proof. The proof is analogous to that of Lemma 8.12. $\square$

Theorem 8.19 is now a straightforward consequence of Lemma 8.21.

Proof of Theorem 8.19. We prove that the REACHABILITY problem for reliable counter machines is reducible to the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$.
Let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, with $\ell \in Q - H$, and define

$$\psi_{\mathcal{M}} \ := \ \mathsf{grid}^{fin} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}} \wedge \Box_h^+\big(\Box_h\bot \to \Box_v^+(c \to \widehat{S}_\ell)\big).$$

We show that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a computation of $\mathcal{M}$ in which ℓ occurs.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$ for some model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ is a finite linear order and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

By Lemma 8.20 there are two sequences $\langle a_k \in W_h : k < L\rangle$ and $\langle b_k \in W_v : k < L\rangle$, of length $L \leq \omega$, such that $\mathfrak{M}, (a_k, b_k) \models \Box_h\bot$ if and only $k + 1 = L$. Since $\mathcal{C}$ comprises only finite linear orders must have that $L < \omega$ is finite, and consequently, $\mathfrak{M}, (a_m, r_v) \models \Box_h\bot$, for $m = L - 1$.

By Lemma 8.21, there is a some computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L'\rangle$ of $\mathcal{M}$, of length $L' > L$, such that, for all $k < L$,

$$\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}.$$

In particular, since $\mathfrak{M}, (r_h, r_v) \models \Box_h^+\big(\Box_h\bot \to \Box_v^+(c \to \widehat{S}_\ell)\big)$, we must have that $\mathfrak{M}, (a_m, b_m) \models \widehat{S}_\ell$. Hence there is some computation of $\mathcal{M}$ in which ℓ occurs.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has a computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ such that $q_m = q_\ell$ for some $m < \omega$. We define the model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (m, <) \in \mathcal{C}$ is a finite strict linear order and $\mathfrak{F}_v = (m, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, by taking $\mathfrak{V}$ to be the valuation defined in the proof of Theorem 8.8, restricted to the domain of $\mathfrak{F}_h \times \mathfrak{F}_v$.

Again, it is straightforward to check that $\mathfrak{M}, (0, 0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable, as required.

Now, since the REACHABILITY problem for reliable counter machines is Σ_1^0-hard, so too must be satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. Hence the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ must be Π_1^0-hard, as required. $\Box$

Moreover, as mentioned above, it follows from Proposition 8.18 that the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is co-recursively enumerable, whenever $\mathcal{C}$ comprises a recursive set of finite frames. Consequently we find that the lower bound given in Theorem 8.19 is optimal, whenever $\mathcal{C}$ is recursive.

Corollary 8.22. *Let $\mathcal{C}$ be the class of all finite strict linear orders. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Π^0_1-complete.*

8.3.4 Dense Linear Orders

Missing from our treatment of products of linear orders, thus far, are results pertaining to those logics having only *reflexive* or *dense* linear orders among their frames, such as **S4.3** or $\mathsf{Log}(\mathbb{Q}, <)$. Such logics are clearly outside of the remit of Theorem 8.8 owing to their omission of the frame $(\omega, <)$, which is neither reflexive nor dense.

Here we employ a version of the *'checker-board' trick* of [120, 121, 102] to extend the results of the above sections to reflexive and dense linear orders.

Theorem 8.23 (Hampson-Kurucz [59])**.** *Let $\mathcal{C}$ be any class of linear orders such that $(\mathbb{O}, <) \in \mathcal{C}$ or $(\mathbb{O}, \leq) \in \mathcal{C}$, for some $\mathbb{O} \in \{\omega, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Σ^0_1-hard.*

Let $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$ be a rooted product model, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ is an arbitrary linear order and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.
Furthermore, let $b \in \textsc{prop}$ be a propositional variable, and define the formula

$$\mathsf{Tick} \;:=\; \Box_v^+ \Box_h^+ (\Diamond_v^+ b \to \Box_v^+ b).$$

The intuition here, is that if $\mathfrak{M}, (r_h, r_v) \models \mathsf{Tick}$, then the valuation $\mathfrak{V}(b)$ forms stripes of 'colour', allowing us to emulate discrete transitions as we alternate between colours, even though $\mathfrak{F}_h$ itself may be dense.

We define an equivalence relation $\sim$ on W_h such that, for all $x, x' \in W_h$,

$$x \sim x' \qquad \Longleftrightarrow \qquad \forall y \in W_v \; \forall z \in \big([x, x'] \cup [x', x]\big);\; (x, y) \in \mathfrak{V}(b) \text{ iff } (z, y) \in \mathfrak{V}(b),$$

where $[x, x'] = \{z \in W_h : xR_h^+ z \text{ and } zR_h^+ x'\}$ denotes the closed interval of points lying between x and x'.

This is to say that two points are $\sim$-equivalent if they occur within the same interval, uniformly coloured with either b or $\neg b$. Let $[x] \subseteq W_h$ denote the unique *uniformly coloured interval* containing $x \in W_h$ and, for each $\varphi \in \mathcal{ML}_2$, write $\mathfrak{M}, ([x], y) \models \varphi$, whenever $\mathfrak{M}, (z, y) \models \varphi$, for all $z \sim x$.

We define a new binary relation $R_h^{\mathfrak{M}}$ over W_h, by taking

$$xR_h^{\mathfrak{M}}y \qquad \iff \qquad xR_hy \text{ and } x \not\sim y,$$

for all $x, y \in W_h$. It is straightforward to verify that $R_h^{\mathfrak{M}}$ retains the transitivity of R_h. However $R_h^{\mathfrak{M}}$ need not be weakly-connected; satisfying, instead, the following weaker property that,

$$\forall x \forall y \forall z \, \big(xR_h^{\mathfrak{M}}y \wedge xR_h^{\mathfrak{M}}z \to (y \sim z \vee yR_h^{\mathfrak{M}}z \vee zR_h^{\mathfrak{M}}y)\big). \tag{8.21}$$

We introduce the following abbreviation

$$\blacklozenge_h\varphi \ := \ \big(b \wedge \Diamond_h(\neg b \wedge \Diamond_h^+\varphi)\big) \vee \big(\neg b \wedge \Diamond_h(b \wedge \Diamond_h^+\varphi)\big),$$

with the property that, for all $\varphi \in \mathcal{ML}_2$,

$$\mathfrak{M}, (x, y) \models \blacklozenge_h\varphi \qquad \iff \qquad \exists x' \in W_h;\ xR_h^{\mathfrak{M}}x' \text{ and } \mathfrak{M}, (x', y) \models \varphi,$$

as can be easily verified.

In place of grid^{fw}, which is clearly unsatisfiable over both dense and reflexive frames, we define the following variation:

$$\mathsf{grid}^{\dagger} \ := \ c \wedge \Box_h^+\Diamond_v(d \wedge \blacklozenge_h c \wedge \neg\blacklozenge_h\blacklozenge_h c).$$

The effect of which is analogous to that described in Lemma 8.9, however here we stipulate, not that each a_k be the immediate R_h-successor of a_{k-1}, but that each a_k be an immediate $R_h^{\mathfrak{M}}$-successor of a_{k-1}.

Lemma 8.24. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{Tick} \wedge \mathsf{grid}^{\dagger}$. Then there exist two infinite sequences $\langle a_k \in W_h : k < \omega\rangle$ and $\langle b_k \in W_v : k < \omega\rangle$ such that, for all $k < \omega$:*

(i) *$r_hR_ha_k$ and $r_vR_v^+b_k$,*

(ii) *If* $k > 0$ *then* a_k *is an immediate* $R_h^{\mathfrak{M}}$*-successor of* a_{k-1},

(iii) $\mathfrak{M}, (a_k, b_k) \models c$,

(iv) *If* $k > 0$ *then* $\mathfrak{M}, (a_k, b_{k-1}) \models d$.

Proof. The proof is completely analogous to that of Lemma 8.9. $\square$

Given a propositional variable $p \in \textsc{prop}$, we define $\mathsf{interval}(p)$ to be the conjunction of the following formulas:

$$\Box_h^+ \Box_v^+ (p \to \blacklozenge_h p' \wedge \neg \blacklozenge_h \blacklozenge_h p'), \tag{8.22}$$

$$\Box_h^+ \Box_v^+ (\blacklozenge_h p' \wedge \neg \blacklozenge_h \blacklozenge_h p' \to p), \tag{8.23}$$

$$\Box_h^+ \Box_v^+ (p \to \neg \blacklozenge_h p), \tag{8.24}$$

where p' is a fresh *auxiliary* variable. The purpose of which is to specify that the variable p, where satisfied, must be satisfied uniformly across the whole interval in which it is situated.

Lemma 8.25. *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{Tick} \wedge \mathsf{interval}(p)$*, and that* $\mathfrak{M}, (x, y) \models p$*, for some* $x \in W_h$*,* $y \in W_v$ *such that* $r_h R^+ x$*, and* $r_v R_v^+ y$*. Then* $\mathfrak{M}, (z, y) \models p$ *if and only if* $z \sim x$*, for all* $z \in W_h$*.*

Proof. Suppose that $x \in W_h$ and $y \in W_v$ are as described. Then by (8.22), we have that $\mathfrak{M}, (x, y) \models \blacklozenge_h p' \wedge \neg \blacklozenge_h \blacklozenge_h p'$. Hence, there is some $x' \in W_h$ such that $x R_h^{\mathfrak{M}} x'$ and $\mathfrak{M}, (x', y) \models p'$. Moreover, we must have that x' is the immediate $R_h^{\mathfrak{M}}$-successor of x, since $\mathfrak{M}, (x, y) \models \neg \blacklozenge_h \blacklozenge_h p'$. Now suppose that $z \sim x$. Then x' is also an immediate $R_h^{\mathfrak{M}}$ of z, and we have that $\mathfrak{M}, (z, y) \models \blacklozenge_h p' \wedge \neg \blacklozenge_h \blacklozenge_h p'$. Hence, it follows from (8.23) that $\mathfrak{M}, (z, y) \models p$, as required.

Conversely, suppose that $z \not\sim x$. Then by (8.21), we have that $x R_h^{\mathfrak{M}} z$ or $z R_h^{\mathfrak{M}} x$. It then follows from (8.24) that $\mathfrak{M}, (z, y) \not\models p$, as required. $\square$

As in Section 8.3.1, we introduce fresh propositional variables p_i and q_i, for each $i < n$. However, here we define

$$\Sigma_i(x) = \{y \in W_v : r_v R_v^+ y \text{ and } \mathfrak{M}, ([x], y) \models p_i \wedge \neg q_i\},$$

for all $x \in W_h$. That is to say, we are concerned with the satisfiability of $p_i \wedge \neg q_i$ across the whole of the interval $[x]$, rather than simply at a single point x, as was the case with the proof of Theorem 8.8.

For each $p \in \{p_i, q_i : i < n\}$, we have the following variation of $\mathsf{counter}(p)$, taking $\mathsf{counter}^\dagger(p)$ to be the conjunction of the following formulas:

$$\mathsf{interval}(\mathsf{start}(p)), \tag{8.25}$$

$$\Box_h^+ \Box_v^+ \big(\mathsf{start}(p) \leftrightarrow (\neg p \wedge \blacksquare_h p)\big), \tag{8.26}$$

$$\Box_h^+ \Box_v^+ (p \to \blacksquare_h p). \tag{8.27}$$

The effect of which is illustrated by the following analogue of Lemma 8.10.

Lemma 8.26. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{Tick} \wedge \mathsf{counter}^\dagger(p)$ and let $x, x' \in W_h$ and $y \in W_v$ be such that $r_h R_h x$, $r_v R_v y$ and x' is the immediate $R_h^{\mathfrak{M}}$-successor of x'. Then we have that:*

(i) *If $\mathfrak{M}, (x, y) \models \mathsf{start}(p)$ then $\mathfrak{M}, ([x], y) \models \neg p$ and $\mathfrak{M}, ([x'], y) \models p$,*

(ii) *If $\mathfrak{M}, (x, y) \models \neg\mathsf{start}(p)$ then $\mathfrak{M}, ([x], y) \models p$ if and only if $\mathfrak{M}, ([x'], y) \models p$.*

Proof. (i) Suppose that $\mathfrak{M}, (x, y) \models \mathsf{start}(p)$. Then by Lemma 8.25, we have that $\mathfrak{M}, ([x], y) \models \mathsf{start}(p)$. Hence by (8.26), we have that $\mathfrak{M}, ([x], y) \models \neg p \wedge \blacksquare_h p$. It now follows that $\mathfrak{M}, ([x'], y) \models p$, since $x R_h^{\mathfrak{M}} x'$, as required.

(ii) Suppose that $\mathfrak{M}, (x, y) \models \neg\mathsf{start}(p)$. Then again by Lemma 8.25, we must have that $\mathfrak{M}, ([x], y) \models \neg\mathsf{start}(p)$. Hence by (8.26), we have that $\mathfrak{M}, ([x], y) \models \blacksquare_h p \to p$. Suppose that $\mathfrak{M}, ([x'], y) \models p$. Then by (8.27), we have that $\mathfrak{M}, ([x], y) \models \blacksquare_h p$, since x' is an immediate $R_h^{\mathfrak{M}}$-successor of x. It then follows that $\mathfrak{M}, ([x], y) \models p$. Conversely, suppose that $\mathfrak{M}, ([x], y) \models p$. Then it is immediate from (8.27) that $\mathfrak{M}, ([x'], y) \models p$, since $x R_h^{\mathfrak{M}} x'$, as required.

$\Box$

Take $\mathsf{counter}^\dagger$ to be the conjunction of $\mathsf{counter}^\dagger(p)$, for all $p \in \{p_i, q_i : i < n\}$, together with (8.12) from above, which stipulates that we cannot mark a counter as 'off' before it is marked as 'on'. For each $i < n$, we define the formulas $\mathsf{fix}_i^{fw}, \mathsf{inc}_i^{fw}$ and dec_i^{fw}, as in (8.13)–(8.15). It is then straightforward to verify that Lemma 8.11 still holds when we replace $\mathsf{grid} \wedge \mathsf{counter}$ with the modified $\mathsf{Tick} \wedge \mathsf{grid}^\dagger \wedge \mathsf{counter}^\dagger$. Indeed, we have no trouble in proving the following analogue of Lemma 8.12, where $\varphi_{\mathcal{M}}$ is taken to be the conjunction of formulas (8.17)–(8.19).

Lemma 8.27 (Emulation Lemma)**.** *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{Tick} \wedge \mathsf{grid}^\dagger \wedge \mathsf{counter}^\dagger \wedge \varphi_{\mathcal{M}}$ and let $\langle a_k \in W_h : k < \omega \rangle$ and $\langle b_k \in W_v : k < \omega \rangle$ be any infinite sequences satisfying conditions (i)–(iv) of Lemma 8.24. Then $\mathcal{M}$ has a non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$ such that $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$, for all $k < \omega$.*

Proof. This proof is identical to that of Lemma 8.12. $\square$

With these modifications in place, we are now able to prove Theorem 8.23.

Proof of Theorem 8.23. We prove that the non-Termination problem for reliable counter machines is reducible to the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$.
To this end, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, and define

$$\psi_{\mathcal{M}} \;:=\; \mathsf{Tick} \wedge \mathsf{grid}^\dagger \wedge \mathsf{counter}^\dagger \wedge \varphi_{\mathcal{M}}.$$

It remains to show that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a non-terminating computation of $\mathcal{M}$.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$ for some model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

It then follows immediately from Lemma 8.27 that $\mathcal{M}$ has a non-terminating computation.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has a reliable non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$. We define a product model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h \in \mathcal{C}$ is one of the frames $(\mathbb{O}, <)$ or $(\mathbb{O}, \leq)$, for some $\mathbb{O} \in \{\omega, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, and $\mathfrak{F}_v = (\omega, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, by taking

$$\begin{aligned}
\mathfrak{V}(b) &= \{(j, m) : m < \omega \text{ and } 2k \leq j < 2k + 1 \text{ for some } k < \omega\},\\
\mathfrak{V}(c) &= \{(k, k) : k < \omega\},\\
\mathfrak{V}(d) &= \{(k, k+1) : k < \omega\},\\
\mathfrak{V}(S_q) &= \{(k, k) : k < \omega \text{ and } q_k = q\}, \quad \text{for each } q \in Q.
\end{aligned}$$

For each $i < n$, we define $\eta_i^+, \eta_i^- : \omega \to \omega$ as in the proof of Theorem 8.8, and take

$$\begin{aligned}\mathfrak{V}(p_i) &= \{(j,m) : m < \eta_i^+(k) \text{ and } k \le j < k+1 \text{ for some } k < \omega\},\\ \mathfrak{V}(q_i) &= \{(j,m) : m < \eta_i^-(k) \text{ and } k \le j < k+1 \text{ for some } k < \omega\},\end{aligned}$$

for all $i < n$. Lastly, as always, we evaluate $\mathsf{start}(p)$ by taking,

$$\mathfrak{V}(\mathsf{start}(p)) = \{(k,m) : (k,m) \notin \mathfrak{V}(p) \text{ and } (k+1,m) \in \mathfrak{V}(p)\},$$

for each $p \in \{p_i, q_i : i < n\}$.

It is then straightforward to check that $\mathfrak{M}, (0,0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable.

Now, since the Termination problem for reliable counter machines is Σ_1^0-hard, so too must be the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$, as required. □

Taking $\mathcal{C}$ to be the class of all reflexive linear orders provides us with a lower bound on the complexity of the decision problem for $\mathbf{S4.3} \times \mathbf{Diff}$, while taking $\mathcal{C}$ to be the class of all right-unbounded, dense linear orders provides us with the lower bound on the decision problem for $\mathsf{Log}(\mathbb{Q}, <) \times \mathbf{Diff}$. Matching upper bounds are provided by Theorem 3.5, since both $\mathsf{Fr}\,\mathbf{S4.3}$ and $\mathsf{Fr}\,\mathsf{Log}(\mathbb{Q}, <)$ are first-order definable.

Corollary 8.28. *The decision problems for both* $\mathbf{S4.3} \times \mathbf{Diff}$ *and* $\mathsf{Log}(\mathbb{Q}, <) \times \mathbf{Diff}$ *are* Σ_1^0*-complete.*

These results stand in marked contrast to the modest complexity for both $\mathbf{S4.3} \times \mathbf{S5}$ and $\mathsf{Log}(\mathbb{Q}, <) \times \mathbf{S5}$, whose decision problems are both decidable in 2ExpTime [38, Theorem 6.61].

Similarly, we may apply these techniques to yield a proof of the following theorem.

Theorem 8.29. *Let $\mathcal{C}$ be any class of discrete linear orders such that $(\mathbb{O}, \leq) \in \mathcal{C}$, for some $\mathbb{O} \in \{\omega, \mathbb{Z}\}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Π^1_1-hard.*

Proof. Let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, with $\ell \in Q - H$, and define

$$\psi_{\mathcal{M}} \;:=\; \mathsf{Tick} \wedge \mathsf{grid}^{\dagger} \wedge \mathsf{counter}^{\dagger} \wedge \varphi_{\mathcal{M}} \wedge \Box_h^+ \blacklozenge_h \Box_v^+ (c \rightarrow \widehat{S}_{\ell}).$$

It follows, in direct analogue to Theorem 8.14, that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a computation of $\mathcal{M}$ in which ℓ occurs infinitely often

Now, since the BÜCHI problem for counter machines is Σ^1_1-hard, so too must be the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. Hence, the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ must be Π^1_1-hard, as required. $\Box$

Consequently, we obtain a similar non-analytic lower bound for the decision problem of $\mathbf{S4.3Dum} \times \mathbf{Diff}$, as was proved for $\mathbf{DisK4.3} \times \mathbf{Diff}$.

Corollary 8.30. *The decision problem for $\mathbf{S4.3Dum} \times \mathbf{Diff}$ is Π^1_1-hard.*

8.3.5 Noetherian Linear Orders

A linear order $\mathfrak{F} = (W, R)$ is said to be *Noetherian* if it contains *no* infinite strictly ascending chains. This, of course, is a specialisation of the modal discreteness that we considered in Section 8.3.2. Examples of Noetherian linear orders include the natural numbers under their reverse ordering $(\omega, >)$, as well as all finite frames, which we previously discussed in Section 8.3.3.

It was proved by Segerberg [112] that an arbitrary strict linear order is Noetherian if and only if it validates *Löb's axiom* (originating in [80]):

$$(l\ddot{o}b) \;:=\; \Box(\Box p \rightarrow p) \rightarrow \Box p,$$

while an arbitrary reflexive linear order is Noetherian if and only if it validates *Grzegorczyk's axiom* (originating in [53]):

$$(grz) \;:=\; \Box\big(\Box(p \rightarrow \Box p) \rightarrow p\big) \rightarrow p.$$

We define **GL.3** to be the smallest normal extension of **K4.3** containing (*löb*), whose frames comprise the class of all Noetherian strict linear orders[†], while **Grz.3** is defined to be the smallest normal extension of **S4.3** containing (*grz*), whose frames comprise the class of all Noetherian *reflexive* linear orders.

The immediate obstacle to our previous approach is that we cannot hope to encode an infinite computation along the forward direction, as both **GL.3** and **Grz.3** interdict all infinite strictly ascending R_h-chains that were crucial to our proofs of Theorems 8.8 and 8.14.

In this section, we address this problems with the aid of a new *'backwards'* grid construction, whose infinitely descending R_h-chains make for a suitable replacement of those given above.

Theorem 8.31 (Hampson-Kurucz [59]). *Let $\mathcal{C}$ be any class of Noetherian strict linear orders such that $(\omega+1, >) \in \mathcal{C}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is Π^1_1-hard.*

Let $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$ be a product model, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ is a Noetherian strict linear order and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

Here we take propositional variables c, d and r, and define grid^{bw} be to the conjunction of the following formulas:

$$\Diamond_h(c \wedge \Box_h \bot), \tag{8.28}$$

$$\Box_h \Diamond_v(r \wedge \neg \Diamond_h^+ c), \tag{8.29}$$

$$\Box_v(\Diamond_h r \to \Diamond_h c), \tag{8.30}$$

$$\Box_h\big(\Diamond_h \top \to \Diamond_v(d \wedge \Diamond_h c \wedge \neg \Diamond_h \Diamond_h c)\big). \tag{8.31}$$

These formulas, when taken together, impose upon $\mathfrak{M}$ the existence of an infinite 'backwards-looking' grid, an example of which is illustrated in Figure 8.2. The grid is constructed from infinitely many finite piece-wise 'forward-looking' grids that are connected in sequence. Indeed, note that conjunct (8.29), responsible to generating these finite piece-wise 'forward-looking' grids, is precisely that given in (8.20) of Section 8.3.3. The following lemmas makes these notions precise.

[†]Thus named for its association with Gödel's provability logic [119]. To be precise $\mathbf{GL} := \mathbf{K4} + (\textit{löb})$ is the *Godel-Löb logic*, of which $\mathbf{GL.3} := \mathbf{GL} + (.3)$ is a normal extension.

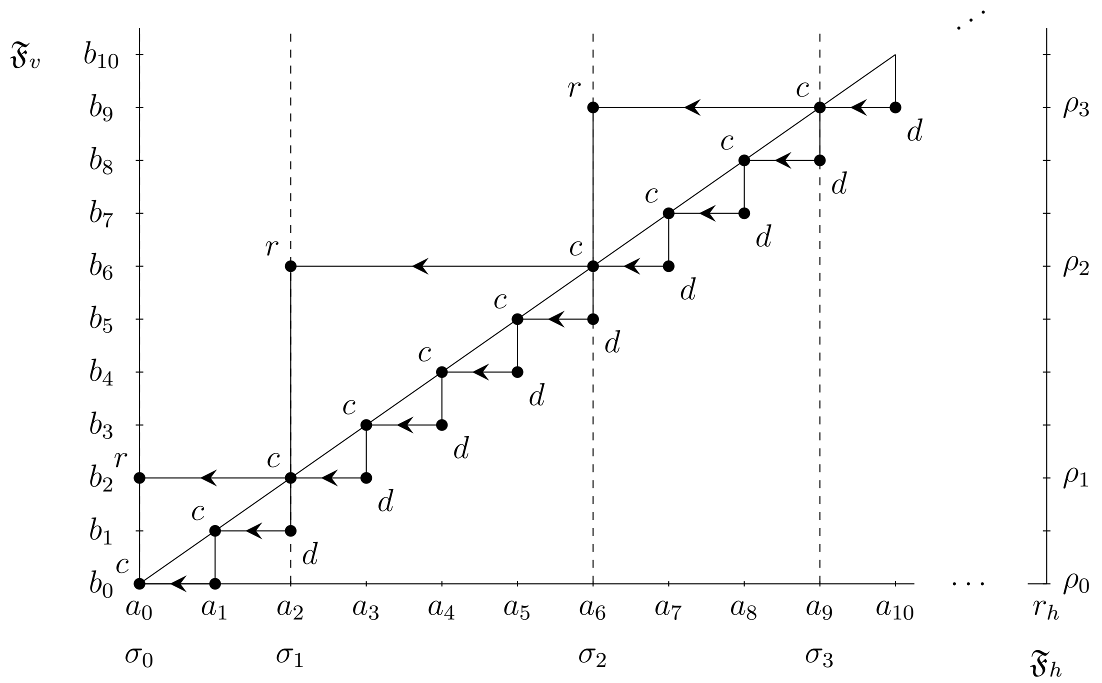


Figure 8.2: Illustration of a possible grid generated by grid^{bw}.

Lemma 8.32. *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{bw}$. *Then there are two infinite sequences* $\langle \sigma_k \in W_h : k < \omega \rangle$ *and* $\langle \rho_k \in W_v : k < \omega \rangle$ *such that* $\mathfrak{M}, (\sigma_0, r_v) \models \Box_h \bot$ *and, for all* $k < \omega$*:*

(i) $r_h R_h \sigma_k$ *and* $r_v R_v^+ \rho_k$*,*

(ii) *If* $k > 0$ *then* $\sigma_k R_h \sigma_{k-1}$*,*

(iii) $\mathfrak{M}, (\sigma_k, \rho_k) \models c$*,*

(iv) *If* $k > 0$ *then* $\mathfrak{M}, (\sigma_{k-1}, \rho_k) \models r$*.*

Proof. First, let $\rho_0 = r_v$ then, by (8.28), there is some $\sigma_0 \in W_h$ such that $r_h R_h \sigma_0$ and $\mathfrak{M}, (\sigma_0, \rho_0) \models c \wedge \Box_h \bot$, as required.

Now suppose that we have already defined $\sigma_k \in W_h$ and $\rho_k \in W_v$, for some $k < \omega$. By (i), we have that $r_h R_h \sigma_k$ and so it follows from (8.29) that there is some $\rho_{k+1} \in W_v$ such that $r_v R_v \rho_{k+1}$ and $\mathfrak{M}, (\sigma_k, \rho_{k+1}) \models r \wedge \Box_h^+ \neg c$. Hence by (8.30) there must be some $\sigma_{k+1} \in W_h$ such that $r_h R_h \sigma_{k+1}$ and $\mathfrak{M}, (\sigma_{k+1}, \rho_{k+1}) \models c \wedge \Box_h \neg r$. Moreover, it follows from the weak-transitivity of $\mathfrak{R}_h$ that $\sigma_{k+1} R_h \sigma_k$ since $\mathfrak{M}, (\sigma_k, \rho_{k+1}) \models \Box_h^+ \neg c$, as required. $\Box$

Note that we do not, here, insist that each σ_k be the *immediate* R_h-successor of σ_{k+1}, for $k < \omega$. However, since $\mathfrak{F}_h$ is Noetherian, there can be at most finitely many R_h-successors separating each σ_k from σ_0, which itself has no R_h-successors. The following lemma echoes Lemma 8.20 in making this statement precise.

Lemma 8.33. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{bw}$. Then for all $m < \omega$ there are two sequences $\langle a^m_k \in W_h : k < L_m \rangle$ and $\langle b^m_k \in W_v : k < L_m \rangle$ of length $L_m \leq \omega$ such that, for all $k < L_m$:*

(i) *$a^m_0 = \sigma_m$ and if $k > 0$ then a^m_k is the immediate R_h-successor of a^m_{k-1},*

(ii) *$b^m_0 = \rho_m$ and if $k > 0$ then $r_v R^+_v b_k$,*

(iii) *$\mathfrak{M}, (a^m_k, b^m_k) \models c$,*

(iv) *If $k > 0$ then $\mathfrak{M}, (a^m_{k-1}, b^m_k) \models d$,*

(v) *$\mathfrak{M}, (a^m_k, r_v) \models \Box_h \bot$ if and only if $k + 1 = L$.*

Proof. The proof is identical to that of Lemma 8.20 and analogous to that of Lemma 8.9 before it, with the addition that the first instance of $a^m_0 \in W_h$ and $b^m_0 \in W_v$ are guaranteed by Lemma 8.32, above. $\square$

By Lemma 8.33(v) we have that $\mathfrak{M}, (a^m_k, r_v) \models \Box_h \bot$ if and only if $k + 1 = L_m$, for all $m < \omega$, while by Lemma 8.32 we find that $\mathfrak{M}, (\sigma_0, r_v) \models \Box_h \bot$. It thus follows from the linearity of R_h that $a^m_k = \sigma_0$ if and only if $k+1 = L_m$. Moreover, since $\mathfrak{F}_h$ is Noetherian and each $\langle a^m_k \in W_h : k < L_m \rangle$ forms an ascending R_h-chain of length L_m, we must have that $L_m < \omega$ is finite for all $m < \omega$. Furthermore, by Lemma 8.32(ii) we have that $\rho_{k+1} R_h \rho_k$ for all $k < \omega$ and thus we must have that $L_{k+1} > L_k$, for all $k < \omega$.

From this, it follows that we may construct two new infinite sequences by concatenating, in reverse order, each of the finite sequences $\langle a^m_k \in W_h : L_{m-1} \leq k < L_m \rangle$ and $\langle b^m_k \in W_v : L_{m-1} \leq k < L_m \rangle$, respectively.

Lemma 8.34. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{bw}$. Then for all $m < \omega$ there are two sequences $\langle a_k \in W_h : k < \omega \rangle$ and $\langle b_k \in W_v : k < \omega \rangle$ such that $\mathfrak{M}, (a_0, b_0) \models \Box_h \bot$ and, for all $k < \omega$:*

(i) *$r_h R_h a_k$ and $r_v R^+_v b_k$,*

(ii) *If* $k > 0$ *then* a_k *is the immediate* R_h*-predecessor of* a_{k-1}*,*

(iii) $\mathfrak{M}, (a_k, b_k) \models c$*,*

(iv) *If* $k > 0$ *then* $\mathfrak{M}, (a_k, b_{k-1}) \models d$*,*

(v) $\mathfrak{M}, (a_k, b_k) \models \Diamond_h r$ *whenever* $k + 1 = L_m$*, for some* $m < \omega$*.*

Proof. For each $m < \omega$, let $\langle a_k^m \in W_h : k < L_m \rangle$ and $\langle b_k^m \in W_v : k < L_m \rangle$ be finite sequences as described by Lemma 8.33. We then define two new infinite sequences $\langle a_k \in W_h : k < \omega \rangle$ and $\langle b_k \in W_v : k < \omega \rangle$, by taking

$$a_k = a_\ell^m \text{ and } b_k = b_\ell^m \qquad \Longleftrightarrow \qquad L_{m-1} \leq k < L_m \quad \text{and} \quad \ell = L_m - k - 1,$$

for all $k < \omega$. That is to say that we concatenate each finite sequences $\langle a_k^m \in W_h : L_{m-1} \leq k < L_m \rangle$ and $\langle b_k^m \in W_v : L_{m-1} \leq k < L_m \rangle$, in reverse order.

It is then immediate from the conditions of Lemmas 8.32 and 8.33, above, that these two infinite sequences satisfy all of the above criteria. □

Crucially, clause Lemma 8.34(v) ensures that $\mathfrak{M}, (a_k, b_k) \models \Diamond_h r$, for infinitely many $k < \omega$. This will be paramount to the following encoding of the BÜCHI problem.

For each $i < n$, we introduce the propositional variables p_i and q_i, as well as the auxiliary variables $\mathsf{start}(p_i)$ and $\mathsf{start}(q_i)$, and define $\mathsf{counter}$ to be the conjunction of equations (8.10)–(8.12). We retain the definition that

$$\Sigma_i(x) = \{y \in W_v : r_v R_v^+ y \text{ and } \mathfrak{M}, (x, y) \models p_i \wedge \neg q_i\},$$

for all $x \in W_h$.

Rather than repeat a similarly lengthy, and otherwise uninsightful, proof for some analogue to Lemma 8.11, we may directly employ the same counting mechanism without modification; albeit under a somewhat different utilisation.

In the proof of Theorem 8.8, we first placed the p_i variables, onto an initially empty row, to mark counter $i < n$ as 'on' before later adding the variable q_i to switch it 'off'. Here we consider this process in reverse by first initialising all rows with both p_i and q_i and first removing q_i to mark counter $i < n$ as 'on' and later removing p_i to switch it 'off'.

Notice that with this strategy we need only swap inc_i^{fw} and dec_i^{fw}, under the maxim: *'an increment (resp. decrement) viewed in reverse is a decrement (resp. increment)'.*

For each of the counter operation $\alpha \in Op_C$ we define Do_α^{bw}, by taking:

$$\mathsf{Do}_\alpha^{bw} \;:=\; \begin{cases} \mathsf{dec}_i^{fw} \wedge \bigwedge_{j\neq i} \mathsf{fix}_j^{fw} & \text{if } \alpha = i^{++}, \\ \mathsf{inc}_i^{fw} \wedge \bigwedge_{j\neq i} \mathsf{fix}_j^{fw} & \text{if } \alpha = i^{--}, \\ \Box_v^+(p_i \to q_i) \wedge \bigwedge_{j=1}^{n} \mathsf{fix}_j^{fw} & \text{if } \alpha = i^{??}. \end{cases}$$

As above, for each state $q \in Q$, we introduce a fresh propositional variables S_q, and take $\varphi_{\mathcal{M}}$ to be the conjunction of the following formulas:

$$\Box_h\Big((c \wedge \Box_v^+\neg d) \to \widehat{S}_{q_{\mathsf{init}}} \;\wedge\; \bigwedge_{i<n} \Box_v^+(p_i \wedge q_i)\Big), \tag{8.32}$$

$$\Box_h \bigwedge_{q\in Q-H} \Big(\Diamond_v^+\big(d \wedge \Diamond_h(c \wedge \widehat{S}_q)\big) \;\to\; \bigvee_{(q,\alpha,q')\in\Delta} \big(\mathsf{Do}_\alpha \wedge \Box_v^+(c \to \widehat{S}_{q'})\big)\Big), \tag{8.33}$$

$$\Box_h\Box_v^+ \bigwedge_{h\in H} \neg\widehat{S}_h, \tag{8.34}$$

where $\widehat{S}_q := S_q \wedge \bigwedge_{q'\neq q} \neg S_{q'}$.

The first conjunct specifies the initial starting configuration $(q_{\mathsf{init}}, \vec{0}) \in \mathrm{Conf}_{\mathcal{M}}$, while the second governs the behaviour of the machine in accordance to the instructions of $\mathcal{M}$. The last conjunct stipulates that our computation be non-terminating. This is more formally addressed by the following lemma.

Lemma 8.35 (Emulation Lemma)**.** *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{bw} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}}$ and let $\langle a_k \in W_h : k < \omega\rangle$ and $\langle b_k \in W_v : k < \omega\rangle$ be any infinite sequences satisfying conditions (i)–(iv) of Lemma 8.34. Then $\mathcal{M}$ has a non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ such that $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$ for all $k < \omega$.*

Proof. We construct, by induction on the length, an infinite sequence of configurations $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ such that $q_0 = q_{\mathsf{init}}$, $v_0 = \vec{0}$, and for all $k < \omega$:

(i) $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$,

(ii) $v_k(i) = |\Sigma_i(a_k)|$, for all $i < n$,

(iii) If $k > 0$ then $(q_{k-1}, v_{k-1}) \xrightarrow{\mathcal{M}} (q_k, v_k)$.

Firstly, by Lemma 8.34 we have that $r_h R_h a_0$, $r_v R_v^+ b_0$, and $\mathfrak{M}, (a_0, b_0) \models c \wedge \Box_v^+ \neg d$. Whence, by (8.32) we have that $\mathfrak{M}, (a_0, b_0) \models \widehat{S}_{q_{\mathsf{init}}}$ and $\mathfrak{M}, (a_0, b_0) \models \Box_v^+(p_i \wedge q_i)$, for all $i < n$. Hence we may take, as our first configuration, the tuple (q_0, v_0), where $q_0 = q_{\mathsf{init}}$ and $v_0 = \vec{0}$, as required.

Now suppose that $(q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}}$ has already been defined, for some $k < \omega$. Then by the induction hypothesis, we have that $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$, while it follows from (8.34) that $q_k \notin H$. Furthermore, by Lemma 8.34, we have that $\mathfrak{M}, (a_{k+1}, r_v) \models \Diamond_v^+\big(d \wedge \Diamond_h(c \wedge \widehat{S}_{q_k})\big)$. Thus, we may infer from (8.33) that

$$\mathfrak{M}, (a_{k+1}, r_v) \models \mathsf{Do}_\alpha \wedge \Box_v^+(c \to \widehat{S}_{q_{k+1}}),$$

for some $(q_k, \alpha, q_{k+1}) \in \Delta$. It then follows from Lemma 8.34 that $\mathfrak{M}, (a_{k+1}, b_{k+1}) \models \widehat{S}_{q_{k+1}}$, thereby satisfying (i).

Next, we define $v_{k+1} : n \to \omega$, by taking

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})|$$

for all $i < n$, thereby satisfying (ii).

It remains to show that $(q_k, v_k) \xrightarrow{\mathcal{M}} (q_{k+1}, v_{k+1})$. So suppose that $i < n$, and consider each of the four following cases, each of which follows from Lemma 8.11 and the induction hypothesis:

– If $\alpha = i^{++}$, then $\mathfrak{M}, (a_{k+1}, r_v) \models \mathsf{dec}_i$. It then follows that

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})| = |\Sigma_i(a_k)| + 1 = v_k(i) + 1.$$

– If $\alpha = i^{--}$, then $\mathfrak{M}, (a_{k+1}, r_v) \models \mathsf{inc}_i$, and so it follows that

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})| = |\Sigma_i(a_k)| - 1 = v_k(i) - 1.$$

– If $\alpha = i^{??}$, then $\mathfrak{M}, (a_{k+1}, r_v) \models \Box_v^+(p_i \to q_i) \wedge \mathsf{fix}_i$, and so it follows that

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})| = 0 \qquad \text{and} \qquad v_{k+1}(i) = |\Sigma_i(a_{k+1})| = |\Sigma_i(a_k)| = v_k(i).$$

– In all other cases where $\alpha \in \{j^{++}, j^{--}, j^{??}\}$ for $j \neq i$, we find that $\mathfrak{M}, (a_k, r_v) \models \mathsf{fix}_i$. It then follows that

$$v_{k+1}(i) = |\Sigma_i(a_{k+1})| = |\Sigma_i(a_k)| = v_k(i).$$

Thus we conclude that $(q_k, v_k) \xrightarrow{\mathcal{M}} (q_{k+1}, v_{k+1})$, thereby satisfying (iii). Hence, by induction on the length of the sequence, we can construct an appropriate non-terminating computation for $\mathcal{M}$, as required. $\square$

We are now in a position to prove Theorem 8.31.

Proof of Theorem 8.31. We prove that the BÜCHI problem for counter machines is reducible to the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$.

To this end, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, with $\ell \in Q - H$, and define

$$\psi_{\mathcal{M}} \ := \ \mathsf{grid}^{bw} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}} \wedge \Box_h \Box_v^+ (c \wedge \Diamond_h r \to \widehat{S}_\ell).$$

We show that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a computation of $\mathcal{M}$ in which ℓ occurs infinitely often.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable. Then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$, for some model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Diff}$.

Since $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{bw}$, we may take $\langle a_k \in W_h : k < \omega \rangle$ and $\langle b_k \in W_v : k < \omega \rangle$ as defined in Lemma 8.34.

By Lemma 8.35, there is a some non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$ of $\mathcal{M}$ such that,

$$\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k},$$

for all $k < \omega$.

Moreover, by Lemma 8.34(v), we have that $\mathfrak{M}, (a_k, b_k) \models c \wedge \Diamond_h r$, for infinitely many $k < \omega$, from which it follows that $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_\ell$, for infinitely many $k < \omega$. Thus, we conclude that there is some non-terminating computation of $\mathcal{M}$ in which ℓ occurs infinitely often.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has a non-terminating computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$ in which ℓ occurs infinitely often. We define a product model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v)$, where $\mathfrak{F}_h = (\omega + 1, >) \in \mathcal{C}$ and $\mathfrak{F}_v = (\omega, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, by taking

$$\begin{aligned}
\mathfrak{V}(c) &= \{(k, k) : k < \omega\}, \\
\mathfrak{V}(d) &= \{(k, k-1) : 1 \leq k < \omega\}, \\
\mathfrak{V}(r) &= \{(k, \eta_k) : k < \omega\}, \\
\mathfrak{V}(S_q) &= \{(k, k) : 0 \geq k < \omega \text{ and } q = q_k\}, \quad \text{for each } q \in Q.
\end{aligned}$$

For each $i < n$, we define the functions $\eta_i^+, \eta_i^- : \omega \to \omega$ as in the proof of Theorem 8.8. However, here we define the valuations of p_i and q_i slightly differently, by taking

$$\begin{aligned}
\mathfrak{V}(p_i) &= \{(k, m) : k < \omega \text{ and } m \geq \eta_i^-(k)\}, \\
\mathfrak{V}(q_i) &= \{(k, m) : k < \omega \text{ and } m \geq \eta_i^+(k)\},
\end{aligned}$$

for all $i < n$. It follows that $m \in \Sigma_i(k)$ if and only if $(k, m) \in \mathfrak{V}(p_i) - \mathfrak{V}(q_i)$, which is to say that $\eta_i^-(k) \leq m < \eta_i^+(k)$. Therefore $|\Sigma_i(k)| = \big(\eta_i^+(k) - \eta_i^-(k)\big) = v_k(i)$.

Lastly, we evaluate $\mathsf{start}(p)$ by taking,

$$\mathfrak{V}(\mathsf{start}(p)) = \big\{(k, m) : (k, m) \notin \mathfrak{V}(p) \text{ and } (k-1, m) \in \mathfrak{V}(p)\big\},$$

for each $p \in \{p_i, q_i : i < n\}$.

It is then straightforward to check that $\mathfrak{M}, (\omega, 0) \models \psi_{\mathcal{M}}$ and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$-satisfiable.

Now, since the Büchi problem for counter machines is Σ_1^1-hard, so too must be the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. Hence, the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ must be Π_1^1-hard, as required. □

It follows immediately that the decision problem for $\mathbf{GL.3} \times \mathbf{Diff}$ is non-analytic; taking $\mathcal{C}$ to be the class of all Noetherian strict linear orders.

As with Theorem 8.8, the techniques of Section 8.3.4 can be applied here to give a similar non-analytic lower bound for those logics characterised by products whose first is some, possibly reflexive, Noetherian linear order. Hence, we have the following corollary.

Corollary 8.36. *The decision problems for both* $\mathbf{GL.3} \times \mathbf{Diff}$ *and* $\mathbf{Grz.3} \times \mathbf{Diff}$ *are* Π^1_1*-hard.*

8.4 Reduction from Diff to K4.3

In this section we provide a polynomial reduction from the decision problem of **Diff** to that of **K4.3**. The existence of such a reduction should come as no great surprise since it is well-established that both decision problems are coNP-complete — [26] and [91], respectively. However, the following reduction can be easily 'lifted' to products to yield a polynomial reduction from the decision problems for $L \times \mathbf{Diff}$ to those of $L \times \mathbf{K4.3}$, whenever L is Kripke complete.

Consequently, the results attained in the previous sections generalise many known results pertaining to products of the from $L \times \mathbf{K4.3}$, whose proofs make strong use of the linearity of the vertical component. In particular, we obtain a alternative proofs for the undecidability of the decision problems for $\mathbf{K}_u \times \mathbf{K4.3}$ [38, Theorem 7.19] and $\mathbf{K4.3} \times \mathbf{K4.3}$ [102], as well as the non-analyticity of $\mathbf{K}_C \times \mathbf{K4.3}$ [38, Theorem 7.19].

One might be tempted to conclude that since **K4.3** and **Diff** both share the same coNP-completeness, there ought to be some liftable model-level reduction from $L \times \mathbf{K4.3}$ to $L \times \mathbf{Diff}$, and therefore supersede the need for the techniques developed above. However, such intuition would be misplaced, since as we saw in Theorem 4.3, the decision problem for $\mathbf{Diff} \times \mathbf{Diff}$ is coNExpTime-complete, while the decision problem for $\mathbf{Diff} \times \mathbf{K4.3}$ is undecidable.

That is to say that products of the form $L \times \mathbf{Diff}$ may often be *genuinely* simpler than their respective **K4.3** counterparts. Hence, in general, there can be no recursive reduction from $L \times \mathbf{K4.3}$ to $L \times \mathbf{Diff}$ — polynomial or otherwise.

Theorem 8.37. *The decision problem for* **Diff** *is polynomially reducible to that of* **K4.3**.

Proof. Suppose that $\varphi \in \mathcal{ML}_1$, and introduce a fresh propositional variable $\widetilde{\psi}$, for each $\psi \in \mathrm{sub}(\varphi)$. We define the translation $(\cdot)^\dagger : \mathrm{sub}(\varphi) \to \mathcal{ML}_1$, by taking

$$p^\dagger = p, \qquad (\neg\psi) = \neg\psi^\dagger, \qquad (\psi_1 \wedge \psi_2)^\dagger = \psi_1^\dagger \wedge \psi_2^\dagger, \qquad (\Diamond\psi)^\dagger = \widetilde{\psi} \vee \Diamond\psi^\dagger,$$

for $p \in$ PROP, and let ζ be the conjunction of the following clauses:

$$\Box^+(\alpha^\dagger \to \Box\widetilde{\alpha}), \tag{8.35}$$

$$\Diamond^+\widetilde{\alpha} \to \Diamond^+\big(\neg\widetilde{\alpha} \wedge \alpha^\dagger\big), \tag{8.36}$$

for all $\alpha \in \mathrm{sub}(\varphi)$.

We claim that $\varphi \in$ **Diff** if and only if $\zeta \to \varphi^\dagger \in$ **K4.3**.

($\Rightarrow$) Suppose that $\zeta \to \varphi^\dagger \notin$ **K4.3**. Then $\mathfrak{M}, r \models \zeta$ and $\mathfrak{M}, r \not\models \varphi^\dagger$ for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, rooted at $r \in W$, where $\mathfrak{F} = (W, <) \in \mathsf{Fr}\,\mathbf{K4.3}$ is a strict linear order.

We define a model $\mathfrak{M}' = (\mathfrak{F}', \mathfrak{V}')$, where $\mathfrak{F}' = (W, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$ is a difference frame, by taking $\mathfrak{V}'(p) = \mathfrak{V}(p)$, for all propositional variables $p \in \mathrm{sub}(\varphi)$.

We prove, by induction on the size of $\psi \in \mathrm{sub}(\varphi)$, that

$$\mathfrak{M}, x \models \psi^\dagger \qquad \Longleftrightarrow \qquad \mathfrak{M}', x \models \psi, \tag{I.H.}$$

for all $x \in W$.

The cases where ψ is a propositional variable or a Boolean combination of simpler formulas follow immediately from the definition of $\mathfrak{V}'$. So suppose that ψ is of the form $\Diamond\alpha$, for some $\alpha \in \mathrm{sub}(\varphi)$, and suppose that $\mathfrak{M}, x \models (\Diamond\alpha)^\dagger$. We have two cases to consider:

– If $\mathfrak{M}, x \models \widetilde{\alpha}$ then $\mathfrak{M}, r \models \Diamond^+\widetilde{\alpha}$, since $\mathfrak{F}$ is rooted at $r \in W$. Whence by (8.36), there is some $y \in W$ such that $r \leq y$ and $\mathfrak{M}, y \models \neg\widetilde{\alpha} \wedge \alpha^\dagger$. By the induction hypothesis, we have that $\mathfrak{M}', y \models \alpha$. Moreover, since $x \in \mathfrak{V}(\widetilde{\alpha})$ and $y \notin \mathfrak{V}(\widetilde{\alpha})$, we must have that $x \neq y$, and thus $\mathfrak{M}', x \models \Diamond\alpha$.

– If $\mathfrak{M}, x \models \Diamond\alpha^\dagger$ then there is some $y \in W$ such that $x < y$ and $\mathfrak{M}, y \models \alpha^\dagger$. In particular, $x \neq y$ since $\mathfrak{F}$ is a strict linear order. By the induction hypothesis we have that $\mathfrak{M}', y \models \alpha$, and thus $\mathfrak{M}', x \models \Diamond\alpha$.

Conversely, suppose that $\mathfrak{M}', x \models \Diamond\alpha$, for some $x \in W$. Then there is some $y \in W$ such that $x \neq y$ and $\mathfrak{M}', y \models \alpha$. By the induction hypothesis, we have that $\mathfrak{M}, y \models \alpha^\dagger$. We then have two cases, depending on whether $x < y$ or $y < x$:

– If $x < y$ then by definition we have that $\mathfrak{M}, x \models \Diamond\alpha^\dagger$.

– Otherwise $y < x$, and so, by (8.35), we have that $\mathfrak{M}, y \models \Box\widetilde{\alpha}$, since $r \leq y$. Whence it follows that $\mathfrak{M}, x \models \widetilde{\alpha}$, since $y < x$.

Together, we have that $\mathfrak{M}, x \models \widetilde{\alpha} \vee \Diamond\alpha^\dagger$, which is to say that $\mathfrak{M}, x \models (\Diamond\alpha)^\dagger$, as required.

Hence we have that $\mathfrak{M}, x \models \psi^\dagger$ if and only if $\mathfrak{M}', x \models \psi$, for all $\psi \in \mathrm{sub}(\varphi)$ and $x \in W$. In particular, we have that $\mathfrak{M}', r \not\models \varphi$, and thus $\varphi \notin \mathbf{Diff}$.

($\Leftarrow$) Suppose that $\varphi \notin \mathbf{Diff}$. Then $\mathfrak{M}, r \not\models \varphi$ for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, where $\mathfrak{F} = (W, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$ is a difference frame.

We define a new frame $\mathfrak{F}' = (W, <) \in \mathsf{Fr}\,\mathbf{K4.3}$ by taking $<$ to be any arbitrary well-order[†] on W such that $r \leq w$, for all $w \in W$. We then define a model $\mathfrak{M}' = (\mathfrak{F}', \mathfrak{V}')$ over $\mathfrak{F}'$, by taking:

– $\mathfrak{V}'(p) = \mathfrak{V}(p)$ for all propositional variables $p \in \mathrm{sub}(\varphi)$,

– $x \in \mathfrak{V}'(\widetilde{\alpha})$ if and only if there is some $y \in W$ such that $y < x$ and $\mathfrak{M}, y \models \alpha$.

We prove, by induction on the size of $\psi \in \mathrm{sub}(\varphi)$, that

$$\mathfrak{M}, x \models \psi^\dagger \qquad \Longleftrightarrow \qquad \mathfrak{M}', x \models \psi, \tag{I.H.}$$

for all $x \in W$.

The cases where ψ is a propositional variable or a Boolean combination of simpler formulas follow immediately from the definition of $\mathfrak{V}'$. So suppose that ψ is of the form $\Diamond\alpha$, for some $\alpha \in \mathrm{sub}(\varphi)$. In which case we have that,

[†]Recall that $(W, <)$ is a *well-order* if every non-empty subset of W has a $<$-minimal element.

$$\begin{aligned}
\mathfrak{M}, x \models \Diamond\alpha &\iff \exists y \in W;\; x \neq y \text{ and } \mathfrak{M}, y \models \alpha,\\
&\iff \exists y \in W;\; y < x \text{ and } \mathfrak{M}, y \models \alpha\\
&\qquad \text{or} \quad \exists y \in W;\; x < y \text{ and } \mathfrak{M}, y \models \alpha,\\
&\iff \exists y \in W;\; y < x \text{ and } \mathfrak{M}, y \models \alpha\\
&\qquad \text{or} \quad \exists y \in W;\; x < y \text{ and } \mathfrak{M}', y \models \alpha^\dagger, \quad \text{by I.H.,}\\
&\iff \mathfrak{M}', x \models \widetilde{\alpha} \text{ or } \mathfrak{M}', x \models \Diamond\alpha^\dagger, \quad \text{by definition,}\\
&\iff \mathfrak{M}', x \models (\Diamond\alpha)^\dagger.
\end{aligned}$$

Whence it follows that $\mathfrak{M}, x \models \psi$ if and only if $\mathfrak{M}', x \models \psi^\dagger$, for all $\psi \in \mathrm{sub}(\varphi)$ and $x \in W$. In particular, we have that $\mathfrak{M}', r \not\models \varphi^\dagger$.

Furthermore, we claim that $\mathfrak{M}', r \models \zeta$.

- For (8.35), suppose that $r \leq y$ and $\mathfrak{M}', y \models \alpha^\dagger$, for some $\alpha \in \mathrm{sub}(\varphi)$. By (I.H.), we have that $\mathfrak{M}, y \models \alpha$. Let $x \in W$ be such that $y < x$. Then by definition $x \in \mathfrak{V}'(\widetilde{\alpha})$, which is to say that $\mathfrak{M}', y \models \Box\widetilde{\alpha}$, as required.
- For (8.36), suppose that $r \leq x$ and $\mathfrak{M}', x \models \widetilde{\alpha}$. By definition, there is some $y \in W$ such that $y < x$ and $\mathfrak{M}, y \models \alpha$. Since $\mathfrak{F}'$ is well-ordered, there is some $<$-minimal $z \in W$ such that $z \notin \mathfrak{V}'(\widetilde{\alpha})$ and $\mathfrak{M}, z \models \alpha$. It follows from (I.H.) that $\mathfrak{M}', r \models \Diamond^+(\neg\widetilde{\alpha} \wedge \alpha^\dagger)$, as required.

Hence it follows that $\mathfrak{M}', r \not\models \zeta \to \varphi^\dagger$, and thus $\zeta \to \varphi^\dagger \notin \mathbf{K4.3}$.

It follows that the decision problem for **Diff** is polynomially reducible to that of **K4.3**, as required. $\square$

As suggested above, this reduction can be easily lifted to yield a polynomial reduction between the decision problems for $L \times \mathbf{Diff}$ and $L \times \mathbf{K4.3}$, whenever L is Kripke complete. The following theorem is a routine extension of Theorem 8.37.

Theorem 8.38. *Let $\mathcal{C}$ be any non-empty class of unimodal frames. Then the decision problem for* $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ *is polynomially reducible to that of* $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{K4.3})$.

Proof. Suppose that $\varphi \in \mathcal{ML}_2$ and introduce a fresh propositional variable $\widetilde{\psi} \in \textsc{prop}$ for all $\psi \in \mathrm{sub}(\varphi)$. We extend the translation given in the proof of Theorem 8.37 by taking

$$p^\dagger = p, \qquad (\neg\psi)^\dagger = \neg\psi^\dagger, \qquad (\psi_1 \wedge \psi_2)^\dagger = \psi_1^\dagger \wedge \psi_2^\dagger,$$
$$(\Diamond_h \psi)^\dagger = \Diamond_h \psi^\dagger, \qquad (\Diamond_v \psi)^\dagger = \widetilde{\psi} \vee \Diamond_v \psi^\dagger,$$

for $p \in \textsc{prop}$. It is then straightforward to prove, in the spirit of Theorem 8.37, that

$$\varphi \in \mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff}) \qquad \Longleftrightarrow \qquad \Box_h^{\leq m} \zeta \to \varphi^\dagger \in \mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{K4.3}),$$

where $m = \mathrm{md}(\varphi)$. □

In particular, it follows that the decision problem for $L \times \mathbf{Diff}$ is polynomially reducible to that of $L \times \mathbf{K4.3}$, whenever L is Kripke complete; taking $\mathcal{C}$ to be the class of all frames for L.

8.5 Discussion

In the previous sections we have explored the use of counter machine reductions to derive lower bounds for a variety of product logics. This novel approach differs from other undecidability results that typically exploit the 'grid-like' nature of the two-dimensional products to encode various Turing machine or tiling problems [38, 83, 102]. With both 'next-time' and 'universal' modalities in both dimensions, such reductions are rather straightforward — see, for example, case of $\mathbf{K}_u \times \mathbf{K}_u$ considered in [38, Theorem 5.37]. However, in lieu of an appropriate 'next-time' operator, it is sometimes possible to employ a version of Cantor's enumeration of the $\omega \times \omega$-plane to encode a grid along an ascending sequence of 'diagonal' points, with pointers emulating the required horizontal and vertical 'next-time' operators [41, 84, 102].

Owing to the lack of structure endemic in those frames for **Diff**, these tricks fail to find obvious application to products of the form $L \times \mathbf{Diff}$. The use of counter machines is therefore attractive, as it appears to require far less structure than is required of those proofs exploiting Turing machines or tiling problems. Furthermore, this new technique makes direct use of the natural grid-like structure of product frames without needing to encode the grid structure as described above.

In Section 8.3.1 we demonstrated the undecidability for a host of logics of the form $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$, where $\mathcal{C}$ comprises some class of linear orders. This is perhaps the most natural setting for reductions of this type, where the horizontal component represent the linear flow of time. Similarly, we may be interested in models where we have *branching* time, represented by those cases where $\mathcal{C}$ comprises any class of transitive frames.

However, while the construction of a grid-aided 'next-time' operator is unproblematic over branching frames, our proposed counting mechanism relies heavily on the linear structure of our models. As such we cannot so readily apply the above techniques to products such as $\mathbf{K4} \times \mathbf{Diff}$ and $\mathbf{S4} \times \mathbf{Diff}$.

Question 8.39. Are the decision problems for either $\mathbf{K4} \times \mathbf{Diff}$ or $\mathbf{S4} \times \mathbf{Diff}$ decidable?

Perhaps the most noteworthy message of this chapter is that, despite the similarities between $\mathbf{S5}$ and $\mathbf{Diff}$, both in terms of their shared coNP-completeness and in terms of the structure of their frames, there is a vast difference in the computational complexity of their products. Table 8.1 below summarises the disparity between their respective products for a variety of cases that we have considered here.

$\times$	**S5**	**Diff**	**K4.3**
K	coNExpTime-complete [83, Thm. 4.5]	decidable $\boxed{?}$ Theorem 6.12	decidable $\boxed{?}$ [38, Thm. 6.40]
K4	coNExpTime-hard in coN2ExpTime [38, Thms. 5.42, 5.28]	$\boxed{\text{decidable?}}$	Σ^0_1-complete [41, Thm. 2]
K4.3	ExpSpace-hard in 2ExpTime [38, Thms. 6.61, 6.64]	Σ^0_1-complete Theorem 8.8	Σ^0_1-complete [102, Thm. 2.2]
Diff	coNExpTime-complete Theorem 4.3	coNExpTime-complete Theorem 4.3	Σ^0_1-complete Theorem 8.8
$\mathsf{Log}(\omega, <)$	ExpSpace-hard† $\boxed{?}$ [63, Thm. 3.1]	Π^1_1-complete Theorem 8.14	Π^1_1-complete [102, Thm 2.4]
GL.3	ExpSpace-hard‡ $\boxed{?}$	Π^1_1-complete Theorem 8.31	Π^1_1-complete [41, Thm. 4]
$\mathbf{K}_u$	in coN2ExpTime $\boxed{?}$ [38, Thm. 6.50]	Σ^0_1-complete Theorem 8.1	Σ^0_1-complete [38, Thm. 7.19]
$\mathbf{PTL}_{\bigcirc\Box}$	ExpSpace-complete [56]	Π^1_1-hard $\boxed{?}$ Theorem 8.5	Π^1_1-hard $\boxed{?}$ [102, Thm. 2.4]

Table 8.1: Respective complexity of products with **S5**, **Diff** and **K4.3**.

†In [38] it is wrongly stated that $\mathsf{Log}(\omega, <) \times \mathbf{S5}$ is ExpSpace-complete, while, in truth, what is proved is that $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{S5})$ is ExpSpace-complete.

‡This result does not appear to be found explicitly in the current literature. However, it can be easily adapted from the techniques of [63].

Part III

Variations on Product Logics

Chapter 9

First-order Modal Logics and Relativised Products

First-order modal logics extend classical first-order logic by introducing additional modal operators, and are notorious for their bad computational behaviour. Furthermore, many *decidable* fragments of first-order logic such as the *monadic fragment* [81], the two-variable fragment (discussed in Section 4.1), and the *guarded fragment* [7] turn out to have undecidable modal extensions — see [38] for references. However, the *one-variable* fragment of many first-order modal logics are relatively tame, in that their decision problems remain decidable [38]. In much the same way that the unimodal logic **S5** can be identified with the one-variable fragment of classical first-order logic, we may interpret two-dimensional modal logics of the form $L \times \mathbf{S5}$ as one-variable fragments of first-order modal logic. Similarly, we may interpret two-dimensional modal logics of the form $L \times \mathbf{Diff}$ as one-variable fragments of first-order modal logics equipped with additional counting quantifiers of the form $\exists_{>m} x$, for $m = 0, 1$.

Owing to philosophical debate as to how we should interpret statements involving both modal operators and first-order quantifiers, investigation into such first-order modal logics has motivated a range of possible semantics, including those in which the domain of interpretation is permitted to either *expand* or *contract*, relative to the direction of modal accessibility relation [36]. This motivates the consideration of *relativised product logics*, characterised by subframes of product frames that, similarly, expand or contract with respect to a given dimension. In this chapter, we consider the decision problems relating to such relativised product logics.

In Section 9.1, we introduce the syntax and semantics for first-order modal logics (with counting quantifiers) and describe the connection that these logics share with two-dimensional modal logics. Motivated by the expanding and contracting nature of first-order Kripke structures, in Section 9.2 we introduce *relativised product logics* as a generalisation on the product construction described in Section 3.2. In Sections 9.4–9.5, we extend the results of Section 8.3 to relativised product logics with decreasing and expanding domains, respectively.

This chapter builds upon results published in [59] with some additional unpublished results — namely, Theorems 9.8 and 9.9. The proofs presented here differ slightly from those appearing in [59], in their use of *incrementing* counter machine problems rather than the *lossy* counter machine problems originally employed.

9.1 Syntax and Semantics

The language $\mathcal{CML}$ extends $\mathcal{C}$, defined in Section 4.1, by the addition of a single unary modal operator $\Diamond$. The *formulas* of $\mathcal{CML}$ are defined according to the grammar:

$$\varphi \ ::= \quad P_j(x_1, \ldots, x_n) \ \mid \ \neg\varphi \ \mid \ (\varphi_1 \wedge \varphi_2) \ \mid \ \exists_{>m} x \ \varphi \ \mid \ \Diamond\varphi$$

where $P_j \in$ PRED is an n-ary predicate symbol, $x_1, \ldots x_n \in Var$ are first-order variables and $m < \omega$. The *size* of each formula is defined to be the number of symbols it comprises, where m is encoded as a binary integer. As above, we write $\exists x \ \varphi := \exists_{>0} x \ \varphi$, and denote by $\mathcal{QML}$ the fragment comprising those formulas whose only quantifiers are of the form $\exists x_i$, for $x_i \in Var$, corresponding to the traditional (counting-free) formulation of *quantified modal logic*. Of the set of all $\mathcal{CML}$-formulas, let $\mathcal{CML}^n$ be the *n-variable fragment*, comprising all those formulas whose only variables are among $x_0, \ldots, x_{n-1}$, for $0 < n < \omega$.

We interpret formulas of $\mathcal{CML}$ in *first-order Kripke models* of the form $\mathfrak{A} = (\mathfrak{F}, \mathcal{D}, \mathcal{I})$, where $\mathfrak{F} = (W, R)$ is a Kripke frame, $\mathcal{D}$ is a function that associates each $w \in W$ with some *first-order domain* D_w, and $\mathcal{I} : W \times \text{PRED} \to \bigcup_{w \in W} D_w$ is an *interpretation* such that $\mathcal{I}(w, P_j) \subseteq D_w^n$ is an n-any relation on D_w, for every n-ary predicate symbol $P_j \in$ PRED.

We say that the model $\mathfrak{A} = (\mathfrak{F}, \mathcal{D}, \mathcal{I})$ is:

- a *constant domain model* if $D_u = D_v$, for all $u, v \in W$,
- an *expanding domain model* if $D_u \subseteq D_v$, for all uRv, and
- a *decreasing domain model* if $D_u \supseteq D_v$, for all uRv.

A *variable assignment* on $\mathfrak{A}$ is a function $h : Var \to \bigcup_{w \in W} D_w$ mapping variables to elements of the combined domain of $\mathfrak{F}$. Given a model $\mathfrak{A} = (\mathfrak{F}, \mathcal{D}, \mathcal{I})$ and a variable assignment h, we define satisfiability in $\mathfrak{A}$ by taking, for all $w \in W$:

$$\begin{aligned}
\mathfrak{A}, w \models^h P_j(x_1, \dots, x_n) \quad &\iff \quad \big(h(x_1), \dots, h(x_n)\big) \in \mathcal{I}(w, P_j), \\
\mathfrak{A}, w \models^h \neg\varphi \quad &\iff \quad \mathfrak{A}, w \not\models^h \varphi, \\
\mathfrak{A}, w \models^h \varphi_1 \wedge \varphi_2 \quad &\iff \quad \mathfrak{A}, w \models^h \varphi_1 \text{ and } \mathfrak{A}, w \models^h \varphi_2, \\
\mathfrak{A}, w \models^h \Diamond\varphi \quad &\iff \quad \exists v \in W;\ wRv \text{ and } \mathfrak{A}, v \models^h \varphi,
\end{aligned}$$

where $P_j \in$ PRED is an n-ary predicate symbol, and

$$\mathfrak{A}, w \models^h \exists_{>m} x\, \varphi \qquad \iff \qquad \big|\{a \in D_w : \mathfrak{A}, w \models^{h(a/x)} \varphi\}\big| > m,$$

for $m < \omega$, where $h(a/x) : Var \to \bigcup_{w \in W} D_w$ is the variable assignment that agrees with h on all variables except x, for which it assigns the value $a \in D_w$. We say that a formula φ is *valid* in $\mathfrak{A}$ if $\mathfrak{A}, w \models^h \varphi$, for all $w \in W$, and every variable assignment $h : Var \to \bigcup_{w \in W} D_w$.

Given a class of unimodal Kripke frames, we define $\mathsf{QLog}(\mathcal{C})$ to be the set of all formulas of $\mathcal{CML}$ that are valid in every constant domain model whose underlying frame $\mathfrak{F}$ belongs to $\mathcal{C}$. Similarly, we denote by $\mathsf{QLog}_{exp}(\mathcal{C})$ (resp. $\mathsf{QLog}_{dec}(\mathcal{C})$) the set of all $\mathcal{CML}$-formulas that are valid if every *expanding* (resp. *decreasing*) domain models whose underlying frame belongs to $\mathcal{C}$.

The *Barcan formula* and its converse, are given by:

$$(bf) \ := \ \Diamond\exists x P(x) \to \exists x \Diamond P(x) \qquad \text{and} \qquad (cbf) \ := \ \exists x \Diamond P(x) \to \Diamond\exists x P(x)$$

respectively. It is straightforward to verify that every decreasing domain model validates (bf), while every expanding domain modal validates (cbf); every constant domain model is trivially both expanding and decreasing, and thus validates both (bf) and (cbf).

9.1.1 Connection with Modal Logics

It is a well-known and trivial exercise to extend the standard translation described in Section 4.3, to yield a translation between $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{S5})$ and the one-variable fragment $\mathsf{QLog}(\mathcal{C}) \cap \mathcal{QML}^1$, for any class of frames $\mathcal{C}$ (see, for example [38]).

Furthermore, this translation maps surjectively onto the first-order fragment $\mathcal{QML}^1$, such that we may identify the above modal logics as syntactic variants of the first-order logics they describe.

With the same effortlessness, we may also extend the translation between **Diff** and the fragment of first-order logic equipped with counting quantifiers, to yield a similar translation between $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ and the fragment $\mathsf{QLog}(\mathcal{C}) \cap \mathcal{CML}^1$. More precisely, for each propositional variable $p_j \in \textsc{prop}$, we associate a monadic predicate symbol $P_j \in \textsc{pred}$ and define the translation $(\cdot)^\dagger : \mathcal{ML}_2 \to \mathcal{CML}^1$, by taking

$$p_j^\dagger = P_j(x), \qquad (\neg\psi)^\dagger = \neg\psi^\dagger, \qquad (\psi_1 \wedge \psi_2)^\dagger = \psi_1^\dagger \wedge \psi_2^\dagger,$$
$$(\Diamond_h \psi)^\dagger = \Diamond \psi^\dagger, \qquad (\Diamond_v \psi)^\dagger = \exists^{\neq} x\ \psi^\dagger,$$

for $p_j \in \textsc{prop}$, where the quantifier $\exists^{\neq} x$ is as defined in (4.4).

Together with this translation, we may define a one-to-one correspondence between product models, based on frames whose second component comprises a difference frames, and first-order Kripke structures with constant domains. Indeed, let $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$ be an arbitrary product model, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}$ and $\mathfrak{F}_v = (W_v, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$. We associate with $\mathfrak{M}$, the first-order Kripke structure $\mathfrak{M}^\star = (\mathfrak{F}_h, \mathcal{D}, \mathcal{I})$, where $\mathcal{D}$ describes a *constant* domain with $\mathcal{D}(u) = W_v$, for all $u \in W_h$, and

$$\mathcal{I}(u, P_j) \ = \ \{v \in W_v : (u, v) \in \mathfrak{V}(p_j)\}$$

for all $u \in W_h$ and $P_j \in \textsc{pred}$. It then follows from a routine induction that

$$\mathfrak{M}, (u, v) \models \varphi \qquad \Longleftrightarrow \qquad \mathfrak{M}^\star, u \models^h \varphi^\dagger,$$

for all $u \in W_h$, $v \in W_v$ and $\varphi \in \mathcal{ML}_2$, where $h(x) = v$.

Furthermore, since $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ is characterised by the class of all product frames in which the second component is a difference frame, it follows that $\varphi \in \mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ if and only if $\varphi^\dagger$ is a theorem of the fragment $\mathsf{QLog}(\mathcal{C}) \cap \mathcal{CML}^1$.

Proposition 9.1. *Let $\mathcal{C}$ be any class of Kripke frames. Then the decision problem for* $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ *is reducible to that of* $\mathsf{QLog}(\mathcal{C}) \cap \mathcal{CML}^1$.

Thus we may reinterpret the results of Chapter 8 as providing lower bounds for various one-variable fragments of first-order modal logics with counting quantifiers. However, this correspondence pertains only to first-order modal logics with *constant domains.* To extend this connection between first-order modal logics and products of modal logics requires us to consider relativised fragments of product logics.

9.2 Relativised Product Logics

The product frames that we have discussed thus far are a special case of the following, more general, construction for attaining two-dimensional structures.

Let $\mathfrak{F} = (W, R)$ be a (unimodal) Kripke frame and let g be a function associating, with each $x \in W$, a (unimodal) frame $g(x) = (W_x, R_x)$. We define the *relativised frame* $\mathfrak{H}_{\mathfrak{F},g}$, by taking

$$\mathfrak{H}_{\mathfrak{F},g} \;:=\; \big(\{(x,y) : x \in W \text{ and } y \in W_x\}, \bar{R}_h, \bar{R}_v\big),$$

where

$$(x,y)\bar{R}_h(x',y') \iff xRx' \text{ and } y = y',$$
$$(x,y)\bar{R}_v(x',y') \iff x = x' \text{ and } yR_xy',$$

for all $x, x' \in W$, $y \in W_x$ and $y' \in W_{x'}$.

Clearly, if we were to take a *constant domain* where $g(x) = g(y) = \mathfrak{G}$ for all $x, y \in W$, then $\mathfrak{H}_{\mathfrak{F},g} = \mathfrak{F} \times \mathfrak{G}$ is equivalent to the regular definition of a product frame, introduced in Section 3.2. Moreover, it should also be clear that every rooted *subframe* of a full product frame can be represented in this way.

Motivated by the aforementioned connection with first-order modal logics, we consider the following special classes of relativised product frames. We say that $\mathfrak{H}_{\mathfrak{F},g}$ is:

– a *full product frame* if $g(x) = g(y)$, for all $x, y \in W$,

– an *expanding product frame* if $g(x)$ is a subframe† of $g(y)$, whenever xRy, and

– a *decreasing product frame* if $g(y)$ is a subframe of $g(x)$, whenever xRy.

†By which we mean that $W_x \subseteq W_y$ and $R_x = R_y \cap (W_x \times W_x)$.

Figure 9.1 provides an illustration of the sort of intuition intended by these definitions.

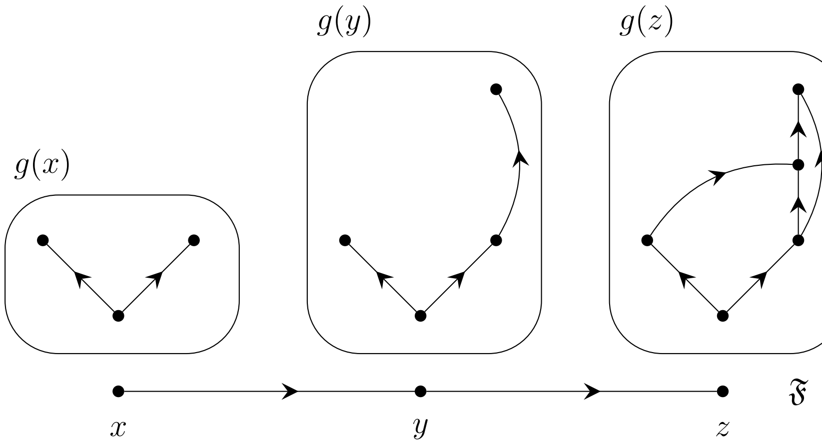


Figure 9.1: Illustration of an expanding product frame.

Given a relativised product frame $\mathfrak{H}_{\mathfrak{F},g}$, we may define a model $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$ — termed an *expanding product model* or *decreasing product model*, as appropriate — by taking $\mathfrak{V} = \langle \mathfrak{V}_x : x \in W \rangle$ to be any collection of propositional valuations $\mathfrak{V}_x : \textsc{prop} \to 2^{W_x}$, for $x \in W$. We define satisfiability in $\mathfrak{M}$ by taking, for all $x \in W$ and $y \in W_x$,

$$
\begin{aligned}
\mathfrak{M}, (x,y) &\models p & &\iff & & y \in \mathfrak{V}_x(p),\\
\mathfrak{M}, (x,y) &\models \neg\varphi & &\iff & & \mathfrak{M}, (x,y) \not\models \varphi,\\
\mathfrak{M}, (x,y) &\models \varphi_1 \wedge \varphi_2 & &\iff & & \mathfrak{M}, (x,y) \models \varphi_1 \text{ and } \mathfrak{M}, (x,y) \models \varphi_2,\\
\mathfrak{M}, (x,y) &\models \Diamond_h\varphi & &\iff & & \exists x' \in W;\ xRx' \text{ and } \mathfrak{M}, (x',y) \models \varphi,\\
\mathfrak{M}, (x,y) &\models \Diamond_v\varphi & &\iff & & \exists y' \in W_x;\ yR_xy' \text{ and } \mathfrak{M}, (x,y') \models \varphi,
\end{aligned}
$$

for $p \in \textsc{prop}$. We say that a φ is *valid* in $\mathfrak{H}_{\mathfrak{F},g}$, if $\mathfrak{M}, (x,y) \models \varphi$, for all $x \in W$ and $y \in W_x$, where $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$ is any model based on $\mathfrak{H}_{\mathfrak{F},g}$.

We define $\mathcal{C}_h \times^{exp} \mathcal{C}_v$ (resp. $\mathcal{C}_h \times^{dec} \mathcal{C}_v$) to be the class of all expanding (resp. decreasing) product frames $\mathfrak{H}_{\mathfrak{F},g}$ such that $\mathfrak{F} = (W, R) \in \mathcal{C}_h$ and $g(w) \in \mathcal{C}_v$, for all $w \in W$. For Kripke complete modal logics L_h and L_v, we write $L_h \times^{exp} L_v$ and $L_h \times^{dec} L_v$ for their respective expanding and decreasing product logics, obtained by taking $\mathcal{C}_i = \mathsf{Fr}\, L_i$, for $i = h, v$. It is known that expanding and decreasing product logics are typically of lower complexity than their full product logic counterparts; indeed in many cases there is a straightforward reduction to full product logics [38, Theorem 9.12].

For our purposes, let $\varphi \in \mathcal{ML}_2$ be an arbitrary formula and take D to be a fresh propositional symbol, not occurring in φ. We define the translation $(\cdot)^D : \mathrm{sub}(\varphi) \to \mathcal{ML}_2$, by taking:

$$p_j^D = p_j, \qquad (\neg\psi)^D = \neg\psi^D, \qquad (\psi_1 \wedge \psi_2)^D = \psi_1^D \wedge \psi_2^D,$$
$$(\Diamond_h \psi)^D = \Diamond_h(D \wedge \psi^D), \qquad (\Diamond_v \psi)^D = \Diamond_v(D \wedge \psi^D),$$

for $p_j \in$ PROP. A straightforward induction on the size of $\varphi \in \mathcal{ML}_2$ proves the following standard result [38, 41].

Theorem 9.2. *Let $\mathcal{C}_h$ and $\mathcal{C}_v$ be any classes of frames, such that $\mathcal{C}_v$ is closed under countable unions*[†]*. Then:*

– $\varphi \in \mathsf{Log}(\mathcal{C}_h \times^{exp} \mathcal{C}_v)$ *if and only if* $D \wedge \Box_h^{\leq m}\Box_v^{\leq m}(D \to \Box_h D) \to \varphi^D \in \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)$,

– $\varphi \in \mathsf{Log}(\mathcal{C}_h \times^{dec} \mathcal{C}_v)$ *if and only if* $D \wedge \Box_v^{\leq m}\Box_h^{\leq m}(\Diamond_h D \to D) \to \varphi^D \in \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)$,

where $m = \mathrm{md}(\varphi)$.

It therefore follows that, under the conditions of Theorem 9.2, the computational complexity of the decision problem for both expanding and decreasing products cannot exceed that of their full product counterparts.

One notable exception to Theorem 9.2 is the class of frames for **GL.3**, which has among its frames $(m, <) \in \mathsf{Fr}\,\mathbf{GL.3}$, for all $m < \omega$, while their union $(\omega, <)$ is markedly absent from $\mathsf{Fr}\,\mathbf{GL.3}$, owing to its infinitely ascending $<$-chain. However, our continuing focus will remain firmly fixed upon those cases where $\mathcal{C}_v = \mathsf{Fr}\,\mathbf{Diff}$, which is readily seen to be closed under countable unions.

Take $(\cdot)^\dagger : \mathcal{ML}_2 \to \mathcal{C}$ to be the translation described above in Section 9.1.1. However, here we must describe a more general — but otherwise, analogous — one-to-one correspondence between *relativised* product models and first-order Kripke structure. Let $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$ be an arbitrary relativised product model, where $\mathfrak{F} = (W, R) \in \mathcal{C}$ and

[†] Given a countable collection of frames $\langle (W_i, R_i) : i \in I \rangle$, we define their *union* to be the frame (W, R), where $W = \bigcup_{i \in I} W_i$ and $R = \bigcup_{i \in I} R_i$.

$g(u) = (W_u, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, for all $u \in W$. We associate with $\mathfrak{M}$, the first-order Kripke structure $\mathfrak{M}^\star = (\mathfrak{F}, \mathcal{D}, \mathcal{I})$, where $\mathcal{D}(u) = W_u$, for all $u \in W$, and

$$\mathcal{I}(u, P_j) \;=\; \mathfrak{V}_u(p_j)$$

for all $u \in W$ and $P_j \in \textsc{pred}$. By a similarly routine induction we find that,

$$\mathfrak{M}, (u, v) \models \varphi \qquad \Longleftrightarrow \qquad \mathfrak{M}^\star, u \models^h \varphi^\dagger,$$

for all $u \in W$, $v \in W_u$ and $\varphi \in \mathcal{ML}_2$, where $h(x) = v$.

Since $\mathfrak{M}^\star$ is a decreasing first-order Kripke model precisely when $\mathfrak{M}$ is a decreasing relativised product model, it follows that $\varphi \in \mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$ if and only if $\varphi^\dagger$ is a theorem of the fragment $\mathsf{QLog}_{dec}(\mathcal{C}) \cap \mathcal{CML}^1$. Similarly, since $\mathfrak{M}^\star$ is an expanding first-order model precisely when $\mathfrak{M}$ is an expanding model, it follows that $\varphi \in \mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ if and only if $\varphi^\dagger$ is a theorem of the fragment $\mathsf{QLog}_{exp}(\mathcal{C}) \cap \mathcal{CML}^1$.

Proposition 9.3. *Let $\mathcal{C}$ be any class of Kripke frames. Then:*

(i) *The decision problem for $\mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$ is reducible to that of $\mathsf{QLog}_{dec}(\mathcal{C}) \cap \mathcal{CML}^1$,*

(ii) *The decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is reducible to that of $\mathsf{QLog}_{exp}(\mathcal{C}) \cap \mathcal{CML}^1$.*

Hence, the results that follow for relativised product logics also provide analogous lower bounds for various first-order modal logics equipped with counting quantifiers, whose domains are permitted to either expand or contract, relative to the direction of the underlying modal accessibility relation.

9.3 Unreliable Counter Machines

9.3.1 Lossy Counter Machines

Lossy counter machines (LCMs) were introduced by Richard Mayr [86] as a weakened version of the reliable counter machines introduced by Minsky [87], and described here in Section 8.1. Unlike their reliable counterparts, lossy counter machines are permitted to spontaneously 'leak' the contents of their counters both immediately prior to, and immediately subsequent to performing each counter operation. Lossy counter machines constitute a simplified version of the much-studied *lossy FIFO-channel systems*, considered in [2, 20, 3, 1].

Here we follow the direction of [107] and introduce LCMs, not as a special class of counter machines, but rather as counter machines whose computations are predicated on a different operational semantics.

More precisely, given a counter machine $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$, we define the *lossiness relation* $\prec$ on the configuration space $\mathrm{Conf}_{\mathcal{M}}$ of $\mathcal{M}$, by taking

$$(q, v) \prec (q', v') \qquad \iff \qquad q = q' \quad \text{and} \quad v(i) \leq v'(i), \quad \text{for all } i < n,$$

for all $(q, v), (q', v') \in \mathrm{Conf}_{\mathcal{M}}$. We then define the *lossy consecution relation* $\xrightarrow{\mathcal{M}\downarrow}$ for $\mathcal{M}$, by taking

$$\sigma_0 \xrightarrow{\mathcal{M}\downarrow} \sigma_1 \qquad \iff \qquad \exists \sigma_0', \sigma_1' \in \mathrm{Conf}_{\mathcal{M}}; \quad \sigma_0 \succ \sigma_0' \xrightarrow{\mathcal{M}} \sigma_1' \succ \sigma_1,$$

for all $\sigma_0, \sigma_1 \in \mathrm{Conf}_{\mathcal{M}}$. The composite effect of which is that $(q, v) \xrightarrow{\mathcal{M}\downarrow} (q', v')$ if and only if there is some $\alpha \in Op_n$ such that $(q, \alpha, q') \in \Delta$ and, for all $i < n$:

- If $\alpha = i^{++}$ then $v'(i) \leq v(i) + 1$,
- If $\alpha = i^{--}$ then $v'(i) \leq v(i) - 1$,
- If $\alpha = i^{??}$ then $v'(i) = 0$,
- If $\alpha \in \{j^{++}, j^{--}, j^{??}\}$, for some $j \neq i$, then $v'(i) \leq v(i)$.

A *lossy computation* of $\mathcal{M}$ is a sequence of configurations $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L \rangle$ of length $L \leq \omega$, such that:(i) $q_0 = q_{\mathsf{init}}$ and $v_0 = \vec{0}$, (ii) if $k > 0$ then $(q_{k-1}, v_{k-1}) \xrightarrow{\mathcal{M}\downarrow} (q_k, v_k)$, and (iii) $q_k \in H$ if and only if $k+1 = L$, for all $k < L$. As above, we say that a lossy computation is *terminating* if its length $L < \omega$ is finite.

We note that, unlike with reliable counter machines, the emptiness-test $i^{??}$ for lossy counter machines poses no precondition that counter i must actually be empty; since we may always incur a lossy error, dumping the entire contents of the counter, immediately prior to the performing such a test.

This results in a substantial weakening of the expressivity of lossy counter machines. Indeed, it was proved by Mayr [86] that both the Reachability and the Termination problems for lossy counter machines are decidable — albeit, not in time bounded by any primitive recursive function [108].

LCM Termination: (Decidable [86], Ackermann-hard [108])

Given a counter machine $\mathcal{M}$, does every *lossy* computation of $\mathcal{M}$ eventually terminate?

LCM Reachability: (Decidable [86], Ackermann-hard [108])

Given a counter machine $\mathcal{M}$, and a state $\ell \in Q - H$, does $\mathcal{M}$ have a *lossy* computation in which ℓ occurs?

The Büchi problem, by contrast, remains undecidable, but at a substantial cost to its former Σ^1_1-completeness.

LCM Büchi: (Σ^0_1-complete [107])

Given a counter machine $\mathcal{M}$, and a state $\ell \in Q - H$, does $\mathcal{M}$ have a *lossy* computation in which ℓ occurs *infinitely often*?

In addition to the Büchi problem, which asks of the existence of a specific computation in which $\ell \in Q - H$ occurs infinitely often, we mentioned the following variation, which asks only for the existences of computations in which $\ell \in Q - H$ occurs arbitrarily often.

LCM ω-Reachability: (Π^0_1-complete [66])

Given a counter machine $\mathcal{M}$, and a state $\ell \in Q - H$, does $\mathcal{M}$ have a *lossy* computation in which ℓ occurs at least n times, for all $n < \omega$?

These results are collated in Table 9.1, below, for reference and comparison.

9.3.2 Incrementing Counter Machines

Somewhat conversely, is the less well-studied notion of *incrementing counter machines (ICMs)* permitted to spontaneously *increase* the value of their counters both immediately prior, and immediately subsequent to performing each counter operation. Incrementing errors† have been considered in the context of both counter machines and their more expressive FIFO-channel systems [29, 92, 93], but have received far less attention that their lossy counterparts.

As with lossy counter machines, we define incrementing counter machines by way of an alternative operational semantics. Given a counter machine $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$, we define the *incrementing consecution relation* $\xrightarrow{\mathcal{M}\uparrow}$ for $\mathcal{M}$, by taking

$$\sigma_0 \xrightarrow{\mathcal{M}\uparrow} \sigma_1 \qquad \iff \qquad \exists \sigma_0', \sigma_1' \in \mathrm{Conf}_{\mathcal{M}}; \quad \sigma_0 \prec \sigma_0' \xrightarrow{\mathcal{M}} \sigma_1' \prec \sigma_1,$$

for all $\sigma_0, \sigma_1 \in \mathrm{Conf}_{\mathcal{M}}$. The composite effect of which is that $(q, v) \xrightarrow{\mathcal{M}\uparrow} (q', v')$ if and only if there is some $\alpha \in Op_n$ such that $(q, \alpha, q') \in \Delta$ and, for all $i < n$:

- If $\alpha = i^{++}$ then $v'(i) \geq v(i) + 1$,
- If $\alpha = i^{--}$ then $v'(i) \geq v(i) - 1$,
- If $\alpha = i^{??}$ then $v(i) = 0$,
- If $\alpha \in \{j^{++}, j^{--}, j^{??}\}$, for some $j \neq i$, then $v'(i) \geq v(i)$.

An *incrementing computation* of $\mathcal{M}$ is a sequence $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L \rangle$ of length $L \leq \omega$, such that: (i) $q_0 = q_{\mathsf{init}}$ and $v_0 = \vec{0}$, (ii) if $k > 0$ then $(q_{k-1}, v_{k-1}) \xrightarrow{\mathcal{M}\uparrow} (q_k, v_k)$, and (iii) $q_k \in H$ if and only if $k + 1 = L$, for all $k < L$. As above, we say that a lossy computation is *terminating* if its length $L < \omega$ is finite.

We note here that, unlike with lossy counter machines, the emptiness-test for incrementing counter machines *does* pose the precondition that the counter actually be empty, but allows the counter to assume any value immediately after the test.

Incrementing counter machines share something of a duality with lossy counter machines. Indeed, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary lossy counter machine, and define

†Sometimes referred to as *insertion errors* in the literature.

its *opposite* $\mathcal{M}^{\mathsf{op}}$ by reversing the direction of each transition, replacing each increments with decrements, and vice versa [93]. Formally, we define $\mathcal{M}^{\mathsf{op}} = (Q, q_{\mathsf{init}}, n, \Delta^{\mathsf{op}}, H^{\mathsf{op}})$, where

$$\begin{aligned}\Delta^{\mathsf{op}} &= \{(q, i^{++}, q') : (q', i^{--}, q) \in \Delta\} \\ &\qquad \cup \{(q, i^{--}, q') : (q', i^{++}, q) \in \Delta\} \\ &\qquad\quad \cup \{(q, i^{??}, q') : (q', i^{??}, q) \in \Delta\}\end{aligned}$$

and $q \in H^{\mathsf{op}}$ if and only if there is no transition $(q, \alpha, q') \in \Delta^{\mathsf{op}}$, for $q' \in Q$ and $\alpha \in Op_n$.

Intuitively, $\mathcal{M}^{\mathsf{op}}$ reverses the 'arrow of time', whereby any lossy errors incurred by $\mathcal{M}$ will appear as incrementing errors incurred by $\mathcal{M}^{\mathsf{op}}$. This observation yields the following result.

Proposition 9.4 (Ouaknine-Worrell [93]). *For all $(q, v), (q', v') \in \mathrm{Conf}_{\mathcal{M}}$,*

$$(q, v) \xrightarrow{\mathcal{M}\downarrow} (q', v') \qquad \Longleftrightarrow \qquad (q', v') \xrightarrow{\mathcal{M}^{\mathsf{op}}\uparrow} (q, v).$$

Furthermore, there is a lossy computation of $\mathcal{M}$ from (q, v) to (q', v') if and only if there is an incrementing computation of $\mathcal{M}^{\mathsf{op}}$ from (q', v') to (q, v).

An immediate effect of this is that the Reachability problem for incrementing counter machines is equivalent to that of lossy counter machines.

However, the crucial difference between ICMs and LCMs is that, unlike with lossy errors whose errors are necessarily bounded by the constraint that all counters hold non-negative values, incrementing errors observe no such restraint. Hence there is not quite the level of duality between ICMs and LCMs that might otherwise be expected, as can be seen by the following complexity results. These results are collated in Table 9.1, below.

ICM Termination: (PrimRec, non-Elementary [18])

Given a counter machine $\mathcal{M}$, does every *incrementing* computation of $\mathcal{M}$ eventually terminate?

ICM Reachability: (Decidable [86], Ackermann-hard [93])

Given a counter machine $\mathcal{M}$, and a state $\ell \in Q$, does $\mathcal{M}$ have an *incrementing* computation in which ℓ occurs?

ICM BÜCHI: (Π^0_1-complete [92, 29])

Given a counter machine $\mathcal{M}$, and a state $\ell \in Q$, does $\mathcal{M}$ have an *incrementing* computation in which ℓ occurs *infinitely often*?

For the sake of completeness, we note that the ω-REACHABILITY problem for ICMs is also Π^0_1-complete; a result that, to the best of the author's knowledge, is absent from the current literature. This will not be used in any of the subsequent results and to prove this here would take us too far afield. A sketch of the proof, however, can be found in Appendix B.

ICM ω-REACHABILITY: (Π^0_1-complete)

Given a counter machine $\mathcal{M}$, and a state $\ell \in Q$, does $\mathcal{M}$ have an *incrementing* computation in which ℓ occurs at least n times, for all $n < \omega$?

	Reliable	*Lossy*	*Incrementing*
REACHABILITY	Σ^0_1-complete [87]	decidable, ACKERMANN-hard [86, 108]	decidable, ACKERMANN-hard [93]
TERMINATION	Σ^0_1-complete [87]	decidable, ACKERMANN-hard [86, 108]	PRIMREC, non-ELEMENTARY [18]
BÜCHI	Σ^1_1-complete [5]	Σ^0_1-complete [107]	Π^0_1-complete [93, 29]
ω-REACHABILITY	$\boxed{?}$	Π^0_1-complete [66]	Π^0_1-complete See Appendix B

Table 9.1: Reachability problems for reliable and unreliable counter machines.

Interestingly, it appears that the ω-REACHABILITY problem for reliable counter machines remains open, and does not follow from the proof of the Σ^1_1-completeness of the BÜCHI problem, given in [5].

9.4 Decreasing Product Models

Theorem 9.2, above, suggests that the complexity of relativised product logics may well be significantly lower than that of their full product logic counterparts. In this section, however, we show that for *decreasing* product logics we often achieve analogous complexity results for logics of the form $\mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$, where $\mathcal{C}$ comprises some class of linear orders.

Indeed, where $\mathcal{C}$ comprises any class of *modally discrete* strict linear orders, we have the following, easily verified reduction from the decision problem for $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ to that of $\mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$.

Proposition 9.5 (Hampson-Kurucz [59])**.** *Let $\mathcal{C}$ be any class of modally discrete strict linear orders. Then*

$$\varphi \in \mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff}) \qquad \Longleftrightarrow \qquad \Box_v^+\Box_h^+(\Diamond_h\top \to \Box_v\Diamond_h\top) \to \varphi \in \mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff}).$$

This already provides us with results for a host of decreasing product logics, analogous to Theorems 8.19, 8.14 and 8.31, since, in each of these cases, $\mathcal{C}$ represents some subclass of modally discrete linear orders.

Theorem 9.6. (i) *Let $\mathcal{C}$ be any class of finite strict linear orders such that $(m, <) \in \mathcal{C}$, for all $m < \omega$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$ is Π_1^0-hard.*

(ii) *Let $\mathcal{C}$ be any class of modally discrete strict linear orders such that $(\omega, <) \in \mathcal{C}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$ is Π_1^1-hard.*

(ii) *Let $\mathcal{C}$ be any class of Noetherian strict linear orders such that $(\omega + 1, >) \in \mathcal{C}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$ is Π_1^1-hard.*

However, an analogue of Theorem 8.8 does not so easily follow. As it stands, grid^{fw}, as defined in (8.8), does not impose the existence of a grid-like structure upon its *decreasing* models, since it is possible that $r_v \notin W_{a_k}$ for some $k < \omega$, whereupon the induction necessarily halts. To guard against this, we define :

$$\mathsf{grid}^{dec} \;:=\; \mathsf{grid}^{fw} \wedge \Box_h^+\Box_v^+\Diamond_h\top, \tag{9.1}$$

for which we may easily prove the following analogue of Lemma 8.9.

Lemma 9.7. *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{dec}$. *Then there are two infinite sequences* $\langle a_k \in W_h : k < \omega \rangle$ *and* $\langle b_k \in W_{a_k} : k < \omega \rangle$, *such that, for all* $k < \omega$:

(i) $a_0 = r_h$ *and if* $k > 0$ *then* a_k *is the immediate successor of* a_{k-1},

(ii) $b_0 = r_v$ *and if* $k > 0$ *then* $r_v R_{a_{k-1}} b_k$,

(iii) $\mathfrak{M}, (a_k, b_k) \models c$,

(iv) *If* $k > 0$ *then* $\mathfrak{M}, (a_{k-1}, b_k) \models d$,

(v) $\{y \in W_{a_k} : r_v R_{a_k} y\} = \{y \in W_{r_h} : r_v R_{r_h} y\}$.

Proof. First let $a_0 = r_h$ and $b_0 = r_v$, so that $\mathfrak{M}, (a_0, b_0) \models c$, as required.

Now suppose that we have already defined $a_k \in W$, $b_k \in W_{a_k}$, for some $k < \omega$. By (i), we have that $r_h R_h^+ a_k$, and so it follows from (8.8) that there is some $b_{k+1} \in W_{a_k}$ such that $r_v R_{a_k} b_{k+1}$ and $\mathfrak{M}, (a_k, b_{k+1}) \models d \wedge \Diamond_h c \wedge \neg \Diamond_h \Diamond_h c$. Hence there must be some $a_{k+1} \in W$ such that $a_k R a_{k+1}$ and $\mathfrak{M}, (a_{k+1}, b_{k+1}) \models c$. Moreover, a_{k+1} must be the immediate R-successor of a_k, since $\mathfrak{M}, (a_k, b_{k+1}) \models \neg \Diamond_h \Diamond_h c$.

Now suppose that $y \in W_{r_h}$ is such that $r_v R_{r_h}^+ y$. By the induction hypothesis we have that $y \in W_{a_k}$ and $r_v R_{a_k}^+ y$. We have that $\mathfrak{M}, (a_k, y) \models \Diamond_h \top$, and so there is some $x \in W$ such that $a_k R x$ and $y \in W_x$. However, since a_{k+1} is the immediate R-successor of a_k, we must have that $y \in W_{a_{k+1}}$ since $\mathfrak{M}$ is a decreasing product model. Hence, by induction on the length, we can construct two appropriate sequences, as required. $\square$

Condition (v) of Lemma 9.7 stipulates that, in addition to the existence of a grid of points, we also maintain a constant domain throughout this grid.

Hence, by merely substituting grid^{fw} for grid^{dec} in the proofs of Theorems 8.8–8.31, we obtain analogous proofs for their decreasing model counterparts; thereby providing alternative proofs for Theorem 9.6, as well as the following theorem, not covered by the scope of Proposition 9.5.

Theorem 9.8. *Let* $\mathcal{C}$ *be any class of strict linear orders such that* $(\omega, <) \in \mathcal{C}$. *Then the decision problem for* $\mathsf{Log}(\mathcal{C} \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$ *is* Σ_1^0*-hard.*

These results are summarised below in Table 9.2

9.5 Expanding Product Models

Expanding products are often markedly less complex than their respective full products, as can be seen from the wealth of results provided by Gabelaia et al. [41]. We should note that, unlike for decreasing products, the tricks employed in Section 9.4 provide us no advantage when faced with expanding products whose domains may spontaneously increase beyond the scope of our formulas. With decreasing products, stipulating that every point has a successor restricts the possibilities for how the domains may decrease. However, without the luxury of a 'backwards' looking modality, the same cannot be mandated for expanding products.

In this section we re-examine our results of Chapter 8 in the context of expanding products, whose domains may spontaneously increase, adding value to our counters beyond our control. These erroneous increments can be modelled with the aid of *incrementing* counter machines, described above.

9.5.1 Linear Orders

In setting up the necessary modifications to the mechanisms described in Section 8.3.1, we first prove the following — albeit, weaker — analogue of Theorem 8.8, with the aid of *incrementing* counter machines. This result is absent from the work of [59], which appeared before the connection with incrementing counter machines was established. The effect of this is that many of the results of this section (Theorem 8.8 excepted) can be proved with the aid of *lossy* counter machines as well. We will discuss this alternative approach as and where is appropriate.

Theorem 9.9. *Let $\mathcal{C}$ be any class of strict linear orders such that $(\omega, <) \in \mathcal{C}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is non-elementary.*

We fix an arbitrary counter machine $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$, and let $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$ be an expanding product model, where $\mathfrak{F} = (W, R) \in \mathcal{C}$ is a strict linear order and $g(x) = (W_x, R_x) \in \mathsf{Fr}\,\mathbf{Diff}$, for all $x \in W$.

Let $c, d \in$ PROP be propositional variables and take grid^{fw} to be the formula given in (8.8), and repeated here for reference:

$$\mathsf{grid}^{fw} \; := \; c \wedge \Box_h^+ \Diamond_v (d \wedge \Diamond_h c \wedge \neg \Diamond_h \Diamond_h c).$$

As in the proof of Theorem 8.8, this formula imposes upon $\mathfrak{M}$ the existence of an infinite grid, as illustrated in Figure 8.1. The following analogue of Lemma 8.9, makes this statement precise.

Lemma 9.10. *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw}$. *Then there are two infinite sequences* $\langle a_k \in W_h : k < \omega\rangle$ *and* $\langle b_k \in W_{a_k} : k < \omega\rangle$*, such that, for all* $k < \omega$*:*

(i) $a_0 = r_h$ *and if* $k > 0$ *then* a_k *is the immediate successor of* a_{k-1}*,*

(ii) $b_0 = r_v$ *and if* $k > 0$ *then* $r_v R_{a_{k-1}} b_k$*,*

(iii) $\mathfrak{M}, (a_k, b_k) \models c$*,*

(iv) *If* $k > 0$ *then* $\mathfrak{M}, (a_{k-1}, b_k) \models d$*.*

Proof. The proof is identical to that of Lemma 8.9. □

For each $i < n$, we introduce the counter variables p_i and q_i, as well as the auxiliary variables $\mathsf{start}(p_i)$ and $\mathsf{start}(q_i)$, and define $\mathsf{counter}$ to be the conjunction of equations (8.10)–(8.12). Furthermore, we have the following analogue of (8.9):

$$\Sigma_i(x) = \{y \in W_x : r_v R_x^+ y \text{ and } \mathfrak{M}, (x, y) \models p_i \wedge \neg q_i\},$$

for all $i < n$.

For the emulation of incrementing counter machines, however, we may relax the definitions of fix_i, inc_i and dec_i; taking instead the formulas:

$$\mathsf{fix}_i^{exp} := \Box_v^+ \neg \mathsf{start}(q_i), \tag{9.2}$$

$$\mathsf{inc}_i^{exp} := \Diamond_v \mathsf{start}(p_i) \wedge \Box_v^+ \neg \mathsf{start}(q_i), \tag{9.3}$$

$$\mathsf{dec}_i^{exp} := \Diamond_v^{=1} \mathsf{start}(q_i), \tag{9.4}$$

for each $i < n$.

These are weaker analogues of the equations (8.13)–(8.15), given in Section 8.3.1. Our motivation here is that we are interested, not in emulating a *reliable* counting mechanism, but instead, allow explicitly the possibility of *incrementing* errors at each operation. The following analogue of Lemma 8.11 reflects these modifications.

Lemma 9.11 (Incrementing Counting Lemma)**.** *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw} \wedge \mathsf{counter}$. *Then for all* $k < \omega$*:*

(i) *If* $\mathfrak{M}, (a_k, r_v) \models \mathsf{fix}_i^{exp}$*, then* $\Sigma_i(a_{k+1}) \supseteq \Sigma_i(a_k)$*,*

(ii) *If* $\mathfrak{M}, (a_k, r_v) \models \mathsf{inc}_i^{exp}$*, then* $\Sigma_i(a_{k+1}) \supseteq \Sigma_i(a_k) \cup \{z\}$*, for some* $z \notin \Sigma_i(a_k)$*,*

(iii) *If* $\mathfrak{M}, (a_k, r_v) \models \mathsf{dec}_i^{exp}$*, then* $\Sigma_i(a_{k+1}) \supseteq \Sigma_i(a_k) - \{z\}$*, for some* $z \in W_{a_k}$*.*

Proof. (i) Suppose $y \in \Sigma_i(a_k)$ then by definition $r_v R^+_{a_k} y$ and $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$. By (8.13), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(q_i)$. It then follows from (8.12) and Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$, since a_{k+1} is the immediate R-successor of a_k. This is to say that $y \in \Sigma_i(a_{k+1})$.

(ii) By (8.14), there is some $z \in W_{a_k}$ such that $r_v R^+_{a_k} z$ and $\mathfrak{M}, (a_k, z) \models \mathsf{start}(p_i) \wedge \neg\mathsf{start}(q_i)$. By Lemma 8.10 we have that $\mathfrak{M}, (a_k, z) \models \neg p_i$ and $\mathfrak{M}, (a_{k+1}, z) \models p_i$, since a_{k+1} is the immediate R-successor of a_k. It follows from (8.12) that $\mathfrak{M}, (a_k, z) \models \neg q_i$ and so, by Lemma 8.10, $\mathfrak{M}, (a_{k+1}, z) \models \neg q_i$. Hence $z \notin \Sigma_i(a_k)$ and $z \in \Sigma_i(a_{k+1})$.

Now suppose $y \in \Sigma_i(a_k)$ then by definition $r_v R^+_{a_k} y$ and $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$. By (8.14), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(q_i)$. It then follows from (8.12) and Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$. This is to say that $y \in \Sigma_i(a_{k+1})$.

(iii) By (8.15), there is some unique $z \in W_{a_k}$ such that $r_v R^+_{a_k} z$ and $\mathfrak{M}, (a_k, z) \models \mathsf{start}(q_i)$. By Lemma 8.10 we have that $\mathfrak{M}, (a_k, z) \models \neg q_i$ and $\mathfrak{M}, (a_{k+1}, z) \models q_i$. It follows that $z \notin \Sigma_i(a_{k+1})$.

Now suppose $y \in \Sigma_i(a_k)$ and $y \neq z$. Then by definition $r_v R^+_{a_k} y$ and $\mathfrak{M}, (a_k, y) \models p_i \wedge \neg q_i$. By (8.15), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(q_i)$, since $y \neq z$. It then follows from (8.12) and Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i \wedge \neg q_i$. This is to say that $y \in \Sigma_i(a_{k+1})$.

$\square$

As above, we specify the action of each counter operation $\alpha \in Op_n$ by the following combination of these atomic actions on each of the individual counter variables:

$$\mathsf{Do}_\alpha^{exp} := \begin{cases} \mathsf{inc}_i^{exp} \wedge \bigwedge_{j \neq i} \mathsf{fix}_j^{exp} & \text{if } \alpha = i^{++}, \\ \mathsf{dec}_i^{exp} \wedge \bigwedge_{j \neq i} \mathsf{fix}_j^{exp} & \text{if } \alpha = i^{--}, \\ \Box_v^+(p_i \to q_i) \wedge \bigwedge_{j < n} \mathsf{fix}_j^{exp} & \text{if } \alpha = i^{??}. \end{cases} \tag{9.5}$$

As above, we introduce a fresh propositional variable S_q, for each control state $q \in Q$, and take $\varphi^{exp}_{\mathcal{M}}$ to be the conjunction of formulas (8.17)–(8.18), together with the appropriate substitution of $\mathsf{Do}^{exp}_{\alpha}$ in place of $\mathsf{Do}^{fw}_{\alpha}$. The following lemma then follows in the same manner as Lemma 8.12.

Lemma 9.12 (Emulation Lemma)**.** *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw} \wedge \mathsf{counter} \wedge \varphi^{exp}_{\mathcal{M}}$ *and let* $\langle a_k \in W_h : k < \omega\rangle$ *and* $\langle b_k \in W_{a_k} : k < \omega\rangle$ *be any infinite sequences satisfying conditions (i)–(iv) of Lemma 9.10. Then* $\mathcal{M}$ *has a non-terminating* incrementing *computation* $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$ *such that* $\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}$*, for all* $k < \omega$*.*

Proof. Analogous to that of Lemma 8.12. □

We are now in a position to prove Theorem 9.9.

Proof of Theorem 9.9. We prove that the non-Termination problem for *incrementing* counter machines is reducible to the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$. To this end, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, and define

$$\psi_{\mathcal{M}} \;:=\; \mathsf{grid}^{fw} \wedge \mathsf{counter} \wedge \varphi^{exp}_{\mathcal{M}},$$

as in the proof of Theorem 8.8.

We show that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a non-terminating incrementing computation of $\mathcal{M}$.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$-satisfiable. Then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$, for some expanding product model $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$, where $\mathfrak{F} = (W, R) \in \mathcal{C}$ is a strict linear order and $g(x) = (W_x, R_x) \in \mathsf{Fr}\,\mathbf{Diff}$, for each $x \in W$.

It then follows immediately from Lemma 9.12 that $\mathcal{M}$ has a non-terminating incrementing computation.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has a non-terminating incrementing computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega\rangle$. We define the expanding product model $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$, where $\mathfrak{F} = (\omega, <) \in \mathcal{C}$ and $g(k) = (\omega, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, for all $k < \omega$.

For each $k < \omega$, we define $\mathfrak{V}_k : \textsc{prop} \to 2^{\omega}$, by taking

$$\mathfrak{V}_k(c) = \{k\}, \qquad \mathfrak{V}_k(d) = \{k+1\}, \qquad \text{and} \qquad \mathfrak{V}_k(S_q) = \begin{cases} \{k\} & \text{if } q_k = q, \\ \emptyset & \text{otherwise,} \end{cases}$$

for each $q \in Q$.

We define, for each $i < n$, the functions $\eta_i^+, \eta_i^- : \omega \to \omega$, by taking $\eta_i^+(0) = \eta_i^-(0) = 0$ and, for all $k < \omega$,

$$\eta_i^+(k+1) = \begin{cases} \eta_i^+(k) + \big(v_{k+1}(i) - v_k(i)\big) & \text{if } v_{k+1}(i) > v_k(i), \\ \eta_i^+(k) & \text{otherwise,} \end{cases}$$

$$\eta_i^-(k+1) = \begin{cases} \eta_i^-(k) + \big(v_k(i) - v_{k+1}(i)\big) & \text{if } v_{k+1}(i) < v_k(i), \\ \eta_i^-(k) & \text{otherwise.} \end{cases}$$

This is to say that that η_i^+ records the cumulative sum of all the increments made to counter $i < n$ — reliable or otherwise — while η_i^- records the cumulative sum of all the decrements made to counter i. It follows from a simple induction that $\big(\eta_i^+(k) - \eta_i^-(k)\big) = v_k(i)$, for all $k < \omega$.

We then define the valuations of p_i and q_i, by taking

$$\mathfrak{V}_k(p_i) = \{m : m < \eta_i^+(k)\} \qquad \text{and} \qquad \mathfrak{V}_k(q_i) = \{m : m < \eta_i^-(k)\},$$

for all $k < \omega$.

Lastly we evaluate $\mathsf{start}(p)$, by taking

$$\mathfrak{V}_k(\mathsf{start}(p)) = \big\{m : m \notin \mathfrak{V}_k(p) \text{ and } m \in \mathfrak{V}_{k+1}(p)\big\},$$

for each $p \in \{p_i, q_i : i < n\}$.

It is then straightforward to check that $\mathfrak{M}, (0,0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$-satisfiable.

Now, since the Termination problem for incrementing counter machines is non-elementary, so too must be the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$, as required. $\square$

It then follows from Theorem 9.9 that the decision problem for $\mathbf{K4.3} \times^{exp} \mathbf{Diff}$ is non-elementary; taking $\mathcal{C}$ to be the class of all strict linear orders.

Corollary 9.13. *The decision problem for* $\mathbf{K4.3} \times^{exp} \mathbf{Diff}$ *is non-elementary.*

This is a notably weaker result than we were able to obtain for $\mathbf{K4.3} \times \mathbf{Diff}$, whose undecidability follows from Theorem 8.8. These shortcomings resulted from our inability to encode a reliable counting mechanism when faced with expanding domains, which may spontaneously increase, adding value to our counters beyond our control.

It is unknown whether $\mathbf{K4.3} \times^{exp} \mathbf{Diff}$ is decidable. However, it is readily seen to be recursively enumerable by a standard variation of Theorem 3.5.

Question 9.14. Is the decision problem for $\mathbf{K4.3} \times^{exp} \mathbf{Diff}$ decidable? Is it PrimRec?

9.5.2 Modally Discrete Linear Orders

We may, as in Section 8.3.2 above, consider the more restricted case where $\mathcal{C}$ comprises some class of *modally discrete* strict linear order containing $(\omega, <)$. However, here, we employ a reduction from the Büchi problem for *incrementing* counter machines, which is known to be Π^0_1-complete (see Table 9.1).

In a previous version [59, Theorem 5.1], before the connection with incrementing counter machines was established, this result was achieved by a similar reduction from the ω-Reachability problem for *lossy* counter machines; also known to be Π^0_1-complete. The proof, however, is more lengthy and less instructive that the one presented here.

Theorem 9.15 (Hampson-Kurucz [59])**.** *Let $\mathcal{C}$ be any class of modally discrete strict linear orders such that $(\omega, <) \in \mathcal{C}$. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is Σ^0_1-hard.*

Proof. We prove that the Büchi problem for *incrementing* counter machines is reducible to the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$. To this end, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, with $\ell \in Q - H$, and define:

$$\psi_{\mathcal{M}} \;:=\; \mathsf{grid} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}} \wedge \Box_h^+ \Diamond_h \Box_v^+ (c \rightarrow \widehat{S}_\ell).$$

It remains to show that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is a computation of $\mathcal{M}$ in which ℓ occurs infinitely often.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$-satisfiable then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$ for some expanding product model $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$, where $\mathfrak{F} = (W, R) \in \mathcal{C}$ is a modally discrete strict linear order and $g(x) = (W_x, R_x) \in \mathsf{Fr}\,\mathbf{Diff}$, for all $x \in W$.

By Lemma 9.12, there is a some non-terminating incrementing computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$ of $\mathcal{M}$ such that, for all $k < \omega$,

$$\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k},$$

where $a_k \in W_h$ and $b_k \in W_v$ are as defined in Lemma 8.9.

Now, for each $m < \omega$ we have that $r_h R^+ a_m$ and so $\mathfrak{M}, (a_m, r_v) \models \Diamond_h \Box_v^+ (c \to \widehat{S}_\ell)$. Therefore, there is some $a \in W$ such that $a_m R a$ and $\mathfrak{M}, (a, r_v) \models \Box_v^+ (c \to \widehat{S}_\ell)$. Furthermore, since $\mathfrak{F} \in \mathcal{C}$ is modally discrete, we have that $a = a_j$, for some $m < j < \omega$. For otherwise, the sequence $\langle a_k : m < k < \omega \rangle$ would form an infinite ascending chain between r_h and a. It then follows from Lemma 9.10 that $\mathfrak{M}, (a_j, b_j) \models \widehat{S}_\ell$. Thus we may conclude that there is an incrementing computation of $\mathcal{M}$ in which ℓ occurs infinitely often, as required.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has an non-terminating incrementing computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$, in which ℓ occurs infinitely often. Take $\mathfrak{M}$ to be the model defined in the proof of Theorem 9.9. It is then straightforward to check that $\mathfrak{M}, (0, 0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\, \mathbf{Diff})$-satisfiable, as required.

Now, since the Büchi problem for ICMs is Π^0_1-hard, so too must be the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\, \mathbf{Diff})$. Hence, the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\, \mathbf{Diff})$ must be Σ^0_1-hard, as required. $\square$

We note that this is a far cry from the lofty Π^1_1-hardness of the decision problems for both the full and decreasing product counterparts — Theorems 8.14 and 9.6, respectively. Is this ultimately a failing of our approach, or is there something fundamentally easier to deciding membership of expanding products, which is absent from their decreasing product counterparts?

The remainder of this section will be concerned with establishing that, indeed, we can do no better than the lower bound given in Theorem 9.15, and that $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\, \mathbf{Diff})$ is indeed recursively enumerable. The result is derived from a reduction to the decision problem for $\mathsf{Log}((\omega, <) \times^{exp} \mathcal{L}_{\mathsf{fin}})$, examined by Konev et al. [66], where $\mathcal{L}_{\mathsf{fin}}$ comprises the class of all *finite* strict linear orders.

Theorem 9.16 (Konev et al. [66]). *The decision problem for* $\mathsf{Log}((\omega, <) \times^{exp} \mathcal{L}_{\mathsf{fin}})$ *is* Σ^0_1*-complete.*

We first require the following lemma, which permits us to assume the following useful assumptions about the size of those models refuting all non-theorems of $\mathsf{Log}(\mathcal{C}\times^{exp}\mathsf{Fr}\,\mathbf{Diff})$.

Lemma 9.17. *Suppose that $\varphi \in \mathcal{ML}_2$ is refuted in some expanding product frame $\mathfrak{H}_{\mathfrak{F},g}$, where $\mathfrak{F} = (m, <)$, for some $m \leq \omega$, and $g(k) = (W_k, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, for all $k < m$. Then φ can be refuted an expanding product frame $\mathfrak{H}_{\mathfrak{F},g'}$, such that $g'(k) = (W'_k, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$ is finite, for all $k < m$. Indeed, where*

$$|W'_k| \leq k \cdot (1 + 2^{n+1}), \tag{9.6}$$

for all $k < m$, where $n = |sub(\varphi)|$ is the size of φ.

Proof. Let $\mathfrak{H}_{\mathfrak{F},g}$, be as described, and suppose that $\mathfrak{M}, (r_h, r_v) \not\models \varphi$ for some expanding product model $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$. We may assume, without any loss of generality, that $r_h = 0$, and since $\mathfrak{H}_{\mathfrak{F},g}$ is expanding, that $r_v \in W_k$, for all $k < m$.

For each $k < m \leq \omega$, we define $T_k \subseteq W_k$ to be the smallest subset closed under the following conditions:

(i) $r_v \in T_k$,

(ii) For all $y \in T_k$ and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}, (k, y) \models \Diamond_v \psi \quad \Longleftrightarrow \quad \exists y' \in T_k;\ y \neq y' \text{ and } \mathfrak{M}, (k, y') \models \psi.$$

It is straightforward to verify that $|T_k| \leq 1 + 2^{n+1}$, for all $k < m$, where $n = |\mathrm{sub}(\varphi)|$ is the size of φ.

We then define, inductively, a sequence $\langle W'_k \subseteq W_k : k < m\rangle$ by taking

$$W'_0 = T_0 \qquad \text{and} \qquad W'_k = W'_{k-1} \cup T_k, \quad \text{for all } 0 < k < m.$$

Clearly $W'_{k-1} \subseteq W'_k$, for each $0 < k < m$, while it is straightforward to verify that each W'_k is a finite set of size $|W'_k| \leq k \cdot (1 + 2^{n+1})$, as prescribed by (9.6).

We define a new expanding product frame $\mathfrak{H}_{\mathfrak{F},g'}$, by taking $g'(k) = (W'_k, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, for all $k < \omega$, and construct a new model $\mathfrak{M}' = (\mathfrak{H}_{\mathfrak{F},g'}, \mathfrak{V}')$, by taking $\mathfrak{V}' = (\mathfrak{V}'_k)_{k<m}$, where $\mathfrak{V}'_k(p) = \mathfrak{V}_k(p) \cap W'_k$, for all $p \in \mathrm{sub}(\varphi)$ and $k < m$.

A straightforward induction reveals that $\mathfrak{M}, (k, y) \models \psi$ if and only if $\mathfrak{M}', (k, y) \models \psi$, for all $\psi \in \mathrm{sub}(\varphi)$, $k < m$ and $y \in W'_k$. In particular, we have that $\mathfrak{M}', (r_h, r_v) \not\models \varphi$, since we are assured, by (i), that $r_v \in T_0 = W'_{r_h}$, as required. $\square$

In particular, it follows that $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is characterised by its expanding frames of *finite vertical height*. This is crucial to the following reduction of the decision problem for $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ to that of $\mathsf{Log}((\omega, <) \times^{exp} \mathcal{L}_{\mathsf{fin}})$.

Theorem 9.18. *The decision problem for* $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ *is recursively enumerable.*

Proof. We prove that the decision problem for $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is polynomially reducible to the recursively enumerable decision problem for $\mathsf{Log}((\omega, <) \times^{exp} \mathcal{L}_{\mathsf{fin}})$.

To this end, suppose that $\varphi \in \mathcal{ML}_2$, and take $\tau : \mathrm{sub}(\varphi) \to \mathcal{ML}_2$ to be the translation given in the proof of Theorem 8.38. Furthermore, take $\zeta \in \mathcal{ML}_2$ to be the conjunction of (8.35)–(8.36), of the same theorem.

It is then straightforward to prove, in the spirit of Theorem 8.37, that

$$\varphi \in \mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff}) \qquad \Longleftrightarrow \qquad \Box_h^+ \zeta \to \tau(\varphi) \in \mathsf{Log}((\omega, <) \times^{exp} \mathcal{L}_{\mathsf{fin}}).$$

Crucially, we depend on the fact that every non-theorem of $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ can be refuted in some expanding product model of finite vertical height. This we are guaranteed by Lemma 9.17.

Now, since the decision problem for $\mathsf{Log}((\omega, <) \times^{exp} \mathcal{L}_{\mathsf{fin}})$ is recursively enumerable, so too must be the decision problem for $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$, as required. $\square$

Thus, we may conclude that the lower bound expressed in Theorem 9.9 is optimal, and that the decision problem for $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is Σ^0_1-complete. This is in contrast to the lofty Π^1_1-hardness of the decision problems for both $\mathsf{Log}((\omega, <) \times \mathsf{Fr}\,\mathbf{Diff})$ and $\mathsf{Log}((\omega, <) \times^{dec} \mathsf{Fr}\,\mathbf{Diff})$, which follow from Theorems 8.14 and 9.6, respectively.

Corollary 9.19. *The decision problem for* $\mathsf{Log}((\omega, <) \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ *is* Σ^0_1*-complete.*

Note that it is still unknown whether this bound is tight for arbitrary classes of modally discrete strict linear orders. Indeed, we do not know whether $\mathsf{Log}(\omega,<) \times^{exp} \mathbf{Diff}$ is recursively enumerable.

Question 9.20. Is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ recursively enumerable for every class $\mathcal{C}$ of modally discrete strict linear orders?

9.5.3 Finite Linear Orders

As with the case of full product logics, the restriction to classes of *finite* linear orders costs us little extra work. In this section we show that the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is non-primitive recursive by a reduction to the REACHABILITY problem for incrementing counter machines, whenever $\mathcal{C}$ comprises the class of all finite linear orders. In [59], this same result is achieved by a slightly different reduction to the REACHABILITY problem for *lossy* counter machines, which shares the same computational complexity.

Theorem 9.21 (Hampson-Kurucz [59])**.** *Let $\mathcal{C}$ be the class of all finite strict linear orders. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is* ACKERMANN*-hard.*

Let grid^{fin} be as defined in (8.20). It follows that if $\mathfrak{M},(r_h,r_v) \models \mathsf{grid}^{fin}$, then there are two sequences $\langle a_k \in W : k < L\rangle$ and $\langle b_k \in W_{a_k} : k < L\rangle$, of length $L \leq \omega$, satisfying all the conditions of Lemma 8.20.

As in the proof of Theorem 8.19, we may take $\varphi^{exp}_{\mathcal{M}}$ to be the conjunction of equations (8.17)–(8.19), with $\mathsf{Do}^{exp}_{\alpha}$ replacing $\mathsf{Do}^{fw}_{\alpha}$. It is then easy to verify the following analogue of Lemma 8.21.

Lemma 9.22. *Suppose that $\mathfrak{M},(r_h,r_v) \models \mathsf{grid}^{fin} \wedge \mathsf{counter} \wedge \varphi^{exp}_{\mathcal{M}}$, and let $\langle a_k \in W_h : k < L\rangle$ and $\langle b_k \in W_v : k < L\rangle$ be any sequence satisfying conditions (i)–(v) of Lemma 8.20, for some $L \leq \omega$. Then $\mathcal{M}$ has a computation $\langle (q_k,v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L'\rangle$ of length $L' > L$, such that $\mathfrak{M},(a_k,b_k) \models \widehat{S}_{q_k}$, for all $k < L$.*

Proof. The proof is analogous to that of Lemma 8.21. $\square$

We may then prove Theorem 9.21.

Proof of Theorem 9.21. We prove that the REACHABILITY problem for incrementing counter machines is reducible to the satisfiability problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$. To this end, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, with $\ell \in Q - H$, and define

$$\psi_{\mathcal{M}} \;:=\; \mathsf{grid}^{fin} \wedge \mathsf{counter} \wedge \varphi_{\mathcal{M}}^{exp} \wedge \Box_h^+\big(\Box_h\bot \to \Box_v^+(c \to \widehat{S}_\ell)\big).$$

We show that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$-satisfiable if and only if there is an incrementing computation of $\mathcal{M}$ in which ℓ occurs.

($\Rightarrow$) Suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$-satisfiable. Then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$, for some expanding product model $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$, where $\mathfrak{F} = (W, R) \in \mathcal{C}$ is a finite linear order and $g(x) = (W_x, R_x) \in \mathsf{Fr}\,\mathbf{Diff}$, for all $x \in W$.

By the appropriate analogue of Lemma 8.9 there are two sequences $\langle a_k \in W_h : k < L\rangle$ and $\langle b_k \in W_{a_k} : k < L\rangle$, of length $L \leq \omega$, such that $\mathfrak{M}, (a_k, b_k) \models \Box_h\bot$ if and only if $k + 1 = L$. Since $\mathcal{C}$ comprises only finite linear orders, we must have that $L < \omega$ is finite, and consequently that $\mathfrak{M}, (a_m, r_v) \models \Box_h\bot$, for $m = L - 1$.

By Lemma 8.12, there is a some incrementing computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L'\rangle$ of $\mathcal{M}$, of length $L' > L$ such that, for all $k < L$,

$$\mathfrak{M}, (a_k, b_k) \models \widehat{S}_{q_k}.$$

Furthermore, we have that $\mathfrak{M}, (r_h, r_v) \models \Box_h^+\big(\Box_h\bot \to \Box_v^+(c \to \widehat{S}_\ell)\big)$, and so it follows that $\mathfrak{M}, (a_m, b_m) \models \widehat{S}_\ell$. Hence there is some incrementing computation of $\mathcal{M}$ in which ℓ occurs.

($\Leftarrow$) Conversely, suppose that $\mathcal{M}$ has an incrementing computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L\rangle$ of length $L \leq \omega$ such that $q_m = q_\ell$ for some $m < L$. We define the model $\mathfrak{M} = (\mathfrak{H}_{\mathfrak{F},g}, \mathfrak{V})$, where $\mathfrak{F} = (m, <) \in \mathcal{C}$ is a finite strict linear order and $g(x) = (\omega, \neq) \in \mathsf{Fr}\,\mathbf{Diff}$, for all $x \in W$, by taking $\mathfrak{V}$ to be the valuation defined in the proof of Theorem 9.9 restricted to the domain of $\mathfrak{H}_{\mathfrak{F},g}$.

Again, it is straightforward to check that $\mathfrak{M}, (0, 0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$-satisfiable.

Now, since the REACHABILITY problem for incrementing counter machines is ACKERMANN-hard, so too must be the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$, as required. $\square$

In [42], it is proved that the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathcal{L}_{\mathsf{fin}})$ is decidable, whenever $\mathcal{C}$ comprises the class of all finite linear orders. Moreover, since every finite strict linear order is isomorphic to $(m, <)$, for some $m < \omega$, it follows from Lemma 9.17 that $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ too is characterised by its expanding frames of finite vertical height. Thus, we are able to leverage the translation of Theorem 8.38, to yield a polynomial reduction from the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ to that of $\mathsf{Log}(\mathcal{C} \times^{exp} \mathcal{L}_{\mathsf{fin}})$, for any class $\mathcal{C}$ of finite linear orders. This provides us a matching upper bound to that given in Theorem 9.21.

Theorem 9.23. *Let $\mathcal{C}$ be the class of all finite linear orders. Then the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$ is decidable.*

Proof. Analogous to that of Theorem 9.18. $\square$

It remains open whether the ACKERMANN-time lower bound provided by Theorem 9.21 is optimal. Indeed, it is worth noting that the proof of decidability given in [42] relies upon *Kruskal's Tree Theorem*, and provides us no explicit upper bound on the computational complexity.

Furthermore, in that same paper the authors prove that the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathcal{L}_{\mathsf{fin}})$ is *not* decidable in ACKERMANN-time, but instead belongs to the class of HYPERACKERMANN-hard problems, owing to a reduction from the reachability problem for lossy FIFO channel systems [106]. However, it is not clear that the techniques of [42] can be extended to provide a similarly lofty HYPERACKERMANN-hard bound for the decision problem of $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$. It is perhaps not unreasonable to suggest that an ACKERMANN-time upper bound may still be found by more direct means.

Question 9.24. What is the complexity of the decision problem for $\mathsf{Log}(\mathcal{C} \times^{exp} \mathsf{Fr}\,\mathbf{Diff})$, when $\mathcal{C}$ comprises the class of all finite linear orders?

9.6 Discussion

The results of Section 9.4–9.5 are summarised below in Table 9.2, together with the corresponding results for regular products, obtained in Section 8.3.

Note that it remains open whether the decision problem for $\mathbf{K4.3} \times^{exp} \mathbf{Diff}$ is decidable, or whether it shares the same complexity as both its constant and decreasing domain counterparts. Furthermore, it is unknown whether the techniques developed in [66] can be adapted for any arbitrary class of modally discrete linear orders $\mathcal{C}$, thereby providing us with recursively enumerable upper bound for $\mathsf{Log}(\omega, <) \times^{exp} \mathbf{Diff}$.

Question 9.25. What is the complexity of the decision problem for $\mathbf{K4.3} \times^{exp} \mathbf{Diff}$?

Question 9.26. Is $\mathsf{Log}(\omega, <) \times^{exp} \mathbf{Diff}$ recursively enumerable?

	$(\omega, <)$	all finite strict linear orders	all strict linear orders	all modally discrete strict linear orders
full products	Π^1_1-complete Theorems 8.14 & 3.5	Π^0_1-complete Theorem 8.19	Σ^0_1-complete Theorems 8.8 & 3.5	Π^1_1-hard $\boxed{?}$ Theorem 8.14
decreasing products	Π^1_1-complete Theorems 9.6 & 9.2	Π^0_1-complete Theorems 9.6 & 9.2	Σ^0_1-complete Theorems 9.8 & 9.2	Π^1_1-hard $\boxed{?}$ Theorem 9.6
expanding products	Σ^0_1-complete Theorems 9.6 & 9.2	decidable $\boxed{?}$ ACKERMANN-hard Theorems 9.21 & 9.23	non-ELEMENTARY, r.e. $\boxed{\text{decidable?}}$ Theorems 9.9 & 3.5	Σ^0_1-hard $\boxed{\text{r.e.?}}$ Theorem 9.6

Table 9.2: Complexity of $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$ for various classes of linear orders.

Chapter 10

Products with a 'Diagonal' Operator

In Section 4.3, we discussed a variation on the standard translation that identifies the bimodal product logic $\mathbf{S5} \times \mathbf{S5}$ with the two-variable, equality-free, (substitution-free) fragment of first-order logic. In this modal setting, equality can be modelled with an additional *'diagonal'* operator δ, interpreted in square product frames as the identity relation. The resulting 3-modal logic more closely resembles the full two-variable fragment of first-order logic with equality; however, while substitution can be emulated with the aid of a diagonal operator, transposition of variables is still lacking. In [111, 115], this 'dimension-joining' modal treatment of first-order equality by way of an additional modal operator is suggested for products of two *arbitrary* modal logics (together with further modal operators used to 'simulate' substitution and transposition).

It is well-noted that the presence or absence of equality in the two-variable fragment of first-order logic has no discernible affect on the coNExpTime-completeness of its validity problem. Hence, one might expect that the addition of a diagonal element to arbitrary product logics is similarly inconsequential. All the more so, given that the decision problem for $\mathbf{K} \times \mathbf{K}$ remains decidable when augmented with additional modal operators simulating substitution and transposition of first-order variables [117].

However, in this chapter we show such intuitions to be misplaced, and that often the introduction of such a diagonal operator can lead to a considerable increase in complexity. We employ a variation on the techniques described above in Chapters 8–9, by first introducing a novel model of unreliable counter machines that, unlike those lossy and incrementing counter machines described in Section 9.3, prove to be *Turing-complete*. The benefit of this new formalism is that it demands far less structure than would be required of encoding comparatively complex *reliable* counter machine problems.

In Section 10.1 we introduce the syntax and semantics for delta products, and in Section 10.2, we describe their connection with regular products. In particular, we show that the global consequence problem for certain product logics can be reduced to the decision problem for their respective delta products. This provides us with undecidable lower bounds for several delta product logics whose delta-free counterparts are decidable. In particular, we show that the decision problems for the delta products $\mathbf{K} \times^{\delta} \mathbf{K}$ and $\mathbf{K} \times^{\delta} \mathbf{K4}$ are undecidable.

In Section 10.3, we introduce the notion of computation by means of *faulty approximations* as a novel variation on unreliable counter machines. Unlike lossy and incrementing counter machines, we show that computation by faulty approximation is *Turing-complete*. In Section 10.4.1, we exploit the greater flexibility of this new formalism to obtain undecidable lower bounds for a host of delta products, using a variation on the techniques described in Chapter 8. Among which, we show that the decision problem for $\mathbf{K} \times^{\delta} \mathbf{S5}$ is undecidable, despite the decidability of both the decision problem and the global consequence problem for $\mathbf{K} \times \mathbf{S5}$.

In Section 10.4.2, we extend this technique to delta products in which the first component is characterised by some class of linear orders. In particular, we show that the decision problems for $\mathbf{K4.3} \times^{\delta} \mathbf{K}$ and $\mathbf{K4.3} \times^{\delta} \mathbf{S5}$ are both undecidable.

Finally, in Section 10.5, we probe the limitations of this approach and show that the delta product $\mathbf{K} \times^{\delta} \mathbf{Alt}$ — lying outside the remit of the aforementioned results — has the exponential product fmp and is thus decidable. The results of this chapter are to be published in [57].

10.1 Syntax and Semantics

We extend the basic bimodal language by an additional nullary modal operator δ, called the *diagonal constant*. The set formulas $\mathcal{ML}_2^{\delta}$ comprises all those strings generated by the following grammar:

$$\varphi \ ::= \quad p_j \ \mid \ \neg\varphi \ \mid \ (\varphi_1 \wedge \varphi_2) \ \mid \ \Diamond_h \varphi \ \mid \ \Diamond_v \varphi \ \mid \ \delta$$

where $p_j \in \textsc{prop}$ is a propositional variable.

As above, let $\mathrm{sub}(\varphi)$ be the set of all subformulas of φ, for all $\varphi \in \mathcal{ML}_2^{\delta}$, and take the *size* of $\varphi \in \mathcal{ML}_2^\delta$ to be the cardinality of $\mathrm{sub}(\varphi)$. The *modal depth* of $\varphi \in \mathcal{ML}_2^\delta$ is defined as above for formulas of the basic modal language with the addition that $\mathrm{md}(\delta) = 0$.

Given two unimodal frames $\mathfrak{F}_h = (W_h, R_h)$ and $\mathfrak{F}_v = (W_v, R_v)$, we define their *delta product frame* to be the 3-frame

$$\mathfrak{F}_h \times^\delta \mathfrak{F}_v \ := \ \big(W_h \times W_v, \bar{R}_h, \bar{R}_v, D\big),$$

where $\bar{R}_h$ and $\bar{R}_v$ are such that, for all $x, x' \in W_h$ and $y, y' \in W_v$,

$$(x,y)\bar{R}_h(x',y') \iff xR_hx' \text{ and } y = y',$$

$$(x,y)\bar{R}_v(x',y') \iff x = x' \text{ and } yR_vy',$$

and D comprises the set of all *diagonal* elements,

$$D = \{(x,y) \in W_h \times W_v \ : \ x = y\}.$$

Note that there can be at most one diagonal element occurring in each row and in each column of the product $\mathfrak{F}_h \times^\delta \mathfrak{F}_v$, as illustrated in Figure 10.1.

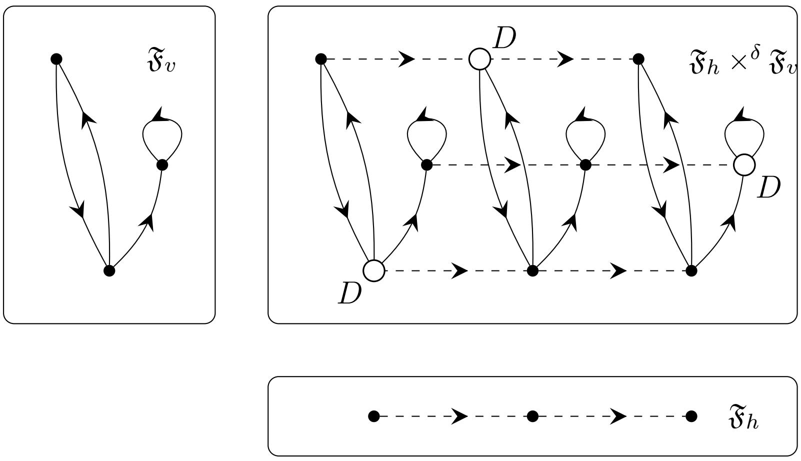


Figure 10.1: Illustration of the delta product construction.

At first, it seems that delta products are somewhat limited in their scope, requiring the domains of their constituent parts to at least overlap. This clearly impacts their applicability to many 'real world' scenarios, such as certain spatio-temporal reasoning in which it makes little sense to speak of a position in space being equal to a position in time. However, since modal languages are incapable of discerning among isomorphic frames, we may instead consider D to be any subset of $W_h \times W_v$ that is both horizontally and vertically 'unique', in the sense that

$$(x, y), (x, y') \in D \implies y = y' \qquad \text{and} \qquad (x, y), (x', y) \in D \implies x = x',$$

for all $x, x' \in W_h$ and $y, y' \in W_v$.

We define modal satisfiability for $\mathcal{ML}_2^\delta$ in much the same way as for regular products, with the additional specification for the atomic formula δ, which is interpreted such that:

$$\mathfrak{M}, (x, y) \models \delta \qquad \iff \qquad (x, y) \in D.$$

For classes $\mathcal{C}_h$ and $\mathcal{C}_v$ of unimodal frames, we define the *delta product class* $\mathcal{C}_h \times^\delta \mathcal{C}_v$ to be the class of all delta products $\mathfrak{F}_h \times^\delta \mathfrak{F}_v$, where $\mathfrak{F}_i \in \mathcal{C}_i$, for $i = h, v$. For modal logics L_h and L_v, we define their *delta product logic* $L_h \times^\delta L_v$, by taking

$$L_h \times^\delta L_v \ := \ \mathsf{Log}(\mathsf{Fr}\, L_h \times^\delta \mathsf{Fr}\, L_v).$$

It is straightforward to check that each delta product logic $L_h \times^\delta L_v$ is a normal extension of its delta-free counterpart $L_h \times L_v$, for all Kripke complete modal logics L_h and L_v. However, this added structure allows for some additional theorems, common to *all* delta product logics. For example, consider the axioms:

$$\begin{aligned}
(hdiag_{k,\ell}) \ &:= \ \Diamond_h^k(\delta \wedge p) \to \Box_h^\ell(\delta \to p), \\
(vdiag_{k,\ell}) \ &:= \ \Diamond_v^k(\delta \wedge p) \to \Box_v^\ell(\delta \to p),
\end{aligned}$$

for $k, \ell < \omega$, which are sound and complete with respect to the class of frames validating the following first-order conditions:

$$\forall x \forall y \big(x R_i^k y \wedge x R_i^\ell z \wedge D(y) \wedge D(z) \to y = z\big),$$

for $i = h, v$. These properties are easily seen to be valid in every delta product frame, and thus $(hdiag_{k,\ell})$ and $(vdiag_{k,\ell})$ join $(com^r), (com^l)$ and (chr) as theorems of every delta product logic.

This additional structure affords delta products a surprising increase in complexity over their delta-free counterparts. In particular, it was proved in [74] that, while $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{K4}$ and $\mathbf{K} \times \mathbf{S5}$ all possess the product fmp, none of the logics $\mathbf{K} \times^{\delta} \mathbf{K}$, $\mathbf{K} \times^{\delta} \mathbf{K4}$ and $\mathbf{K} \times^{\delta} \mathbf{S5}$ enjoy even the *abstract* fmp. However the exact complexity of these delta products remained open.

By contrast, the addition of a diagonal constant adds no further complexity to $\mathbf{S5} \times^{\delta} \mathbf{S5}$, whose decision problem remains coNExpTime-complete, owing to a straightforward variation of the standard translation between $\mathcal{ML}_2^{\delta}$ and the two-variable fragment of first-order logic *with equality*. Similarly, we see that the decision problems for both $\mathbf{Diff} \times^{\delta} \mathbf{Diff}$ and $\mathbf{S5} \times^{\delta} \mathbf{Diff}$ also remain coNExpTime-complete. However, one further discrepancy appears in the case of $\mathbf{S5} \times^{\delta} \mathbf{Diff}$. It was proved in Theorem 6.6 that $\mathbf{S5} \times \mathbf{Diff}$ enjoys the exponential product fmp, while the following analogue of Theorem 6.3 shows that $\mathbf{S5} \times^{\delta} \mathbf{Diff}$ enjoys not even the abstract fmp.

Theorem 10.1. $\mathbf{S5} \times^{\delta} \mathbf{Diff}$ *does not possess the abstract fmp.*

Let $\varphi_{\infty}^{\delta}$ be the conjunction of the following formulas:

$$\Diamond_h \Diamond_v (c \wedge \neg\delta \wedge \Box_h \neg c \wedge \Box_v \neg\delta), \tag{10.1}$$

$$\Box_h \Diamond_v (c \wedge \neg\delta \wedge \Box_h \neg c), \tag{10.2}$$

$$\Box_v \Diamond_h (\delta \wedge \neg c). \tag{10.3}$$

We then have the following analogue of Lemma 6.4.

Lemma 10.2. *Let $\mathfrak{F} = (W, R_h, R_v)$ be any frame for $[\mathbf{S5}, \mathbf{wK5}]$ validating $(hdiag_{1,1})$. If $\varphi_{\infty}^{\delta}$ is satisfiable in $\mathfrak{F}$ then $\mathfrak{F}$ must be infinite.*

Proof. The proof is analogous to that of Lemma 6.4, with the diagonal elements taking the place of the propositional variable d. The vertical uniqueness of each diagonal element is guaranteed by $(hdiag_{1,1})$. $\Box$

In the following chapters, we examine the computational complexity of various delta product logics whose delta-free counterparts are decidable. Among which are the logics $\mathbf{K} \times^{\delta} \mathbf{K}$, $\mathbf{K} \times^{\delta} \mathbf{K4}$ and $\mathbf{K} \times^{\delta} \mathbf{S5}$, whose delta-free counterparts are each characterised by their finite frames and are therefore decidable, as well as the logics $\mathbf{K4.3} \times^{\delta} \mathbf{K}$ and $\mathbf{K4.3} \times^{\delta} \mathbf{S5}$, whose delta-free counterparts do not enjoy the finite model property but are nonetheless decidable.

The undecidability of the first two can be established by a reduction from the *global consequence problem* for $\mathbf{K}\times\mathbf{K}$ and $\mathbf{K}\times\mathbf{K4}$, respectively, which we describe in Section 10.2. The remaining logics are the subject of Section 10.4, where we employ a variation on the counter machine reduction introduced in Chapter 8. However, owing to the reduced expressivity of these logics — being, as they are, unable to express uniqueness of points except for those points appearing along the diagonal — we must first introduce the notion of unreliable computation by way of *faulty approximations*.

10.2 Delta Products and Global Consequences

In this section we show that the decision problem for delta products is related to the problem of deciding *global consequences* for regular product logics. We can therefore, derive many complexity results for delta products from standard results regarding the global consequence problem.

Given an n-modal logic $L \subseteq \mathcal{ML}_n$ and any two modal formulas $\varphi, \psi \in \mathcal{ML}_n$, we say that ψ is a *global L-consequence* of φ, written $\varphi \models^*_L \psi$, if ψ belongs to the smallest set of $\mathcal{ML}_n$ formulas containing $L \cup \{\varphi\}$ that is closed under the inference rules **(MP)** and **(Nec)**, given in Table 2.2. The *global consequence problem* for L asks whether $\varphi \models^*_L \psi$, for a given pair of n-modal formulas $\varphi, \psi \in \mathcal{ML}_n$.

A logic L is said to be *globally Kripke complete* if it is the case that, for all formulas $\varphi, \psi \in \mathcal{ML}_n$, the following conditions are equivalent:

(i) $\varphi \models^*_L \psi$,

(ii) if $\mathfrak{M} \models \varphi$ then $\mathfrak{M} \models \psi$, for every model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ such that $\mathfrak{F} \in \mathsf{Fr}\, L$.

The following result pertains to logics whose frames are closed under the addition of so-called 'spy-points'. Given a unimodal frame $\mathfrak{F} = (W, R)$, we denote by $\mathfrak{F}^\bullet = (W^\bullet, R^\bullet)$ the result of augmenting $\mathfrak{F}$ with an additional *'spy-point'* from which the whole frame can be surveyed, taking

$$W^\bullet = W \cup \{r\} \qquad \text{and} \qquad R^\bullet = R \,\cup\, \{(r, w) : w \in W\}.$$

for some fresh point $r \notin W$. We say that a class of frames $\mathcal{C}$ is closed under the addition of 'spy-points' if $\mathfrak{F}^\bullet \in \mathcal{C}$, whenever $\mathfrak{F} \in \mathcal{C}$. Note that this spy-point technique is well-known in the hybrid logic literature [15].

Proposition 10.3. *Let L_h and L_v be any two Horn-axiomatisable, Kripke complete uni-modal logics such that both $\mathsf{Fr}\, L_h$ and $\mathsf{Fr}\, L_v$ are closed under the addition of 'spy-points', and where $L_h \times L_v$ is globally Kripke complete. Then the global consequence relation for $L_h \times L_v$ is polynomially reducible to the decision problem for $L_h \times^\delta L_v$.*

Proof. Suppose that $\varphi, \psi \in \mathcal{ML}_2$ and let univ^δ be the following formula:

$$\mathsf{univ}^\delta \ := \ \Box_h \Diamond_v \delta \wedge \Box_h \Box_h \Diamond_v \delta \wedge \Box_v \Diamond_h \delta \wedge \Box_v \Box_v \Diamond_h \delta. \tag{10.4}$$

We claim that $\varphi \models^*_{L_h \times L_v} \psi$ if and only if $(\mathsf{univ}^\delta \wedge \Box_h \Box_v \varphi) \to \Box_h \Box_v \psi \in L_h \times^\delta L_v$.

($\Rightarrow$) Suppose that $(\mathsf{univ}^\delta \wedge \Box_h \Box_v \varphi) \to \Box_h \Box_v \psi \notin L_h \times^\delta L_v$. Then $\mathfrak{M}, (r_h, r_v) \models \mathsf{univ}^\delta \wedge \Box_h \Box_v \varphi$ and $\mathfrak{M}, (r_h, r_v) \not\models \Box_h \Box_v \psi$, for some delta product model $\mathfrak{M} = (\mathfrak{F}_h \times^\delta \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_v) \in \mathsf{Fr}\, L_h$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\, L_v$.

Hence, there are $u_h \in W_h$ and $u_v \in W_v$ such that $\mathfrak{M}, (u_h, u_v) \not\models \psi$. Let $\mathfrak{G}_i = (U_i, S_i)$ be the subframe of $\mathfrak{F}_i$ generated by $u_i \in W_i$, for $i = h, v$, and let $\mathfrak{M}' = (\mathfrak{G}_h \times \mathfrak{G}_v, \mathfrak{V})$ be a new model such that $\mathfrak{V}'(p) = \mathfrak{V}(p) \cap (U_h \times U_v)$, for all propositional variables $p \in \textsc{prop}$.

Clearly $\mathfrak{M}', (u_h, u_v) \not\models \psi$, and so it remains to show that $\mathfrak{M}', (x, y) \models \varphi$, for all $x \in U_h$ and $y \in U_v$.

For each $w \in U_i$, let $d_i(w) < \omega$ be the length of the shortest R_i-chain from u_i to w, for $i = h, v$. We prove by induction on the depth $k = d_i(w)$, that

$$r_i R_i w, \text{ for all } w \in U_i \text{ and } i = h, v.$$

Suppose that $i = h$. If $d_h(w) = 0$ then $w = u_h$, in which case we have that $r_h R_h u_h$, by definition. So suppose that $r_h R_h x$, for all $x \in U_h$ of depth $d_h(x) < k$.

Suppose that $w \in U_h$ is such that $d_i(w) = k$. Since $\mathfrak{G}_h$ is rooted at u_i, there is some $w' \in U_h$ such that $d_i(w') < k$ and $w' R_h w$. By the induction hypothesis, we have that $r_h R_h w'$ and so it follows from (10.4) that $\mathfrak{M}, (w, r_v) \models \Diamond_v \delta$. Hence there is some $v \in W_v$ such that $r_v R_v v$ and $\mathfrak{M}, (w, v) \models \delta$; which is to say that $w \in W_v$ and $r_v R_v w$.

It then follows, again from (10.4), that $\mathfrak{M}, (r_h, w) \models \Diamond_h \delta$. Hence there is some $v \in W_h$ such that $r_h R_h v$ and $\mathfrak{M}, (v, w) \models \delta$; which is to say that $r_h R_h w$, as required. The case where $i = v$ is similar.

Since $\mathfrak{M}, (r_h, r_v) \models \Box_h \Box_v \varphi$, we may readily conclude that $\mathfrak{M}', (x, y) \models \varphi$, for all $x \in U_h$ and $y \in U_v$. Hence it follows that $\varphi \not\models^*_{L_h \times L_v} \psi$, as required.

($\Leftarrow$) Suppose that $\varphi \not\models^*_{L_h \times L_v} \psi$. Since $L_h \times L_v$ is assumed to be globally Kripke complete, there is some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ such that $\mathfrak{M} \models \varphi$ and $\mathfrak{M}, (x, y) \not\models \psi$, where $\mathfrak{F}$ is a frame for $L_h \times L_v$. Furthermore, since both L_h and L_v are Horn-axiomatisable, we may suppose — courtesy of Theorem 3.5 — that $\mathfrak{F} = \mathfrak{F}_h \times \mathfrak{F}_v$ is a product frame, where $\mathfrak{F}_i = (W_i, R_i) \in \mathsf{Fr}\, L_i$, for $i = h, v$.

Let $\mathfrak{G}_h = (U_h, S_h)$ be the disjoint union of $|W_v|$-many copies of $\mathfrak{F}_h$, and $\mathfrak{G}_v = (U_v, S_v)$ be the disjoint union of $|W_h|$-many copies of $\mathfrak{F}_v$, where

$$U_h = \{(x, \alpha) : x \in W_h \text{ and } \alpha < |W_v|\} \quad \text{and} \quad U_v = \{(y, \beta) : y \in W_v \text{ and } \beta < |W_h|\}$$

and $(u, k) S_i (v, \ell)$ if and only if $u R_i v$ and $k = \ell$, for $i = h, v$ and $k, \ell < \omega$.

Since $\mathsf{Fr}\, L_i$ is closed under disjoint unions, we have that $\mathfrak{G}_i \in \mathsf{Fr}\, L_i$ and consequently $\mathfrak{G}_i^\bullet \in \mathsf{Fr}\, L_i$, since we further supposed that $\mathsf{Fr}\, L_i$ is closed under the addition of spy-points, for $i = h, v$.

We may now construct a delta product model $\mathfrak{M}' = (\mathfrak{G}_h^\bullet \times^\delta \mathfrak{G}_v^\bullet, \mathfrak{V}')$ by taking

$$\mathfrak{V}'(p) = \big\{\big((x, \alpha), (y, \beta)\big) \ : \ (x, y) \in \mathfrak{V}(p)\big\}$$

for all propositional variables $p \in \textsc{prop}$.

It it straightforward to check that $\mathfrak{M}', (r, r) \models \Box_h \Box_v \varphi$ and $\mathfrak{M}', (r, r) \not\models \Box_h \Box_v \psi$.

Furthermore, since $|U_h| = |U_v|$ we may assume that up to some suitable isomorphism $U_h = U_v$, from which we may establish that $\mathfrak{M}', (r, r) \models \mathsf{univ}^\delta$. Hence it follows that $(\mathsf{univ}^\delta \wedge \Box_h \Box_v \varphi) \to \Box_h \Box_v \psi \notin L_h \times^\delta L_v$, as required.

$\Box$

It is well-established that the logics $\mathbf{K} \times \mathbf{K}$ and $\mathbf{K} \times \mathbf{K4}$ both satisfy the requirements of the Proposition 10.3; indeed, Theorem 5.12 of [38] shows that $L_h \times L_v$ is globally Kripke complete whenever L_h and L_v are both Horn axiomatisable and Kripke complete.

In [83], it is established that the global consequence problem for $\mathbf{K} \times \mathbf{K}$ is undecidable, via a reduction from the unconstrained tiling problem. Furthermore, it is shown in [54] that the reduction of the decision problem for $\mathbf{K4}$ to the global consequence problem for

K, given in [125], can be 'lifted' to yield a similar reduction between the decision problem for $\mathbf{K4} \times \mathbf{K4}$ and the global consequence problem for $\mathbf{K} \times \mathbf{K4}$. However, since the decision problem for $\mathbf{K4}\times\mathbf{K4}$ is known to be undecidable [41], so too must be the global consequence problem for $\mathbf{K} \times \mathbf{K4}$.

It then follows from Proposition 10.3 that the decision problems for $\mathbf{K}\times^{\delta}\mathbf{K}$ and $\mathbf{K}\times^{\delta}\mathbf{K4}$ are both undecidable. However, by a straightforward analogue of Theorem 3.5, both logics can be shown to be recursively enumerable.

Corollary 10.4. *The decision problems for* $\mathbf{K} \times^{\delta} \mathbf{K}$ *and* $\mathbf{K} \times^{\delta} \mathbf{K4}$ *are both* Σ^0_1*-complete.*

It should be noted, that the Proposition 10.3 does not extend to those logics having only weakly-connected frames — such as **K4.3**, **S5** and **Diff** — or those logics having only frames of bounded width, none of which are not closed under the addition of 'spy-points'.

In some of these cases, such a reduction as that provided above is either unhelpful or demonstrably non-existent. For example, while the global consequence problem for $\mathbf{K} \times \mathbf{S5}$ is reducible to the decision problem for $\mathbf{PDL} \times \mathbf{S5}$, and is thus decidable in coN2ExpTime [134, 105], the decision problem for $\mathbf{K} \times^{\delta} \mathbf{S5}$ is shown to be undecidable in Theorem 10.7, below. Furthermore, we show in Section 10.5 that the decision problem for $\mathbf{K} \times^{\delta} \mathbf{Alt}$ can be decided in coNExpTime, whereas the undecidability of the global consequence problem for $\mathbf{K} \times \mathbf{Alt}$ can be established by a straightforward reduction from the unconstrained tiling problem.

10.3 Faulty Approximations

In Section 9.3 we introduced the notion of lossy and incrementing errors, and noted that many decision problems for counter machines become significantly more tractable when we allow for the possibility of spontaneous errors. Indeed, even undecidable problems such as the Reachability and the Termination problems become decidable (albeit, non-elementary) with the addition of such unreliability.

Combinations of lossy and incrementing errors have been considered in the context of FIFO-channel systems [20], however only in the trivial case where both errors afflict the same channel. The result being that Reachability, Termination, and Büchi are all vacuously decidable, as there are no impediments to reachability; we may introduce errors as and when necessary in order to reach any desired configuration.

In this section we introduce the notion of a *faulty approximation* [57] as a means of simulating a reliable computation by means of two distinct sets of counters; one set permitting spontaneous lossy errors with the other permitting spontaneous incrementing errors. These two sets of counters, working in tandem, each compensate for the errors made by the other, thereby maintaining Turing-completeness using only unreliable transitions. This appears to be the first known formulation of an unreliable counter machine that is, nonetheless, Turing-complete.

In the following sections, we exploit the surprising expressivity of such faulty approximations to provide undecidable lower bounds for otherwise seemingly unremarkable delta product logics. Again, we introduce these new counter machines, not as a special class of counter machines, but through the introduction of a new operational semantics.

Definition 10.5. Given a counter machine $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$, we define an *approximant* of $\mathcal{M}$ to be a 3-tuple (q, v^+, v^-) such that $q \in Q$, and $v^+, v^- : n \to \omega$ are upper and lower bounds on the values of the counters of $\mathcal{M}$. The only requirement we enforce is that $v^-(i) \leq v^+(i)$ for all $i < n$, as one would expect. Let $\mathrm{Apx}_{\mathcal{M}}$ be the set of all approximants of $\mathcal{M}$.

We define the *faulty consecution relation* for $\mathcal{M}$, by taking $(q_0, v_0^+, v_0^-) \xrightarrow{\mathcal{M}^{\approx}} (q_1, v_1^+, v_1^-)$ if and only if there is some $\alpha \in Op_n$ such that $(q_0, \alpha, q_1) \in \Delta$ and, for all $i < n$:

- If $\alpha = i^{++}$ then $v_1^+(i) \geq v_0^+(i) + 1$ and $v_1^-(i) \leq v_0^-(i) + 1$,
- If $\alpha = i^{--}$ then $v_1^+(i) \geq v_0^+(i) - 1$ and $v_1^-(i) \leq v_0^-(i) - 1$,
- If $\alpha = i^{??}$ then $v_0^+(i) = 0$, $v_1^+(j) \geq v_0^+(j)$ and $v_1^-(j) \leq v_0^-(j)$,
- If $\alpha \in \{j^{++}, j^{--}, j^{??}\}$ and $j \neq i$ then $v_1^+(j) \geq v_0^+(j)$ and $v_1^-(j) \leq v_0^-(j)$.

A *faulty approximation* of $\mathcal{M}$ is a sequence of approximants $\langle (q_k, v_k^+, v_k^-) \in \mathrm{Apx}_{\mathcal{M}} : k < L \rangle$ of length $L \leq \omega$, such that:

- $q_0 = q_{\mathsf{init}}$ and $v_0^+ = v_0^- = \vec{0}$, where $\vec{0}$ denotes the function which assigns zero to all counters,
- If $k > 0$ then $(q_{k-1}, v_{k-1}^+, v_{k-1}^-) \xrightarrow{\mathcal{M}^{\approx}} (q_k, v_k^+, v_k^-)$,
- $q_k \in H$ if and only if $k + 1 = L$.

for all $k < L$.

Faulty approximations are inherently non-deterministic, combining, as they do, both a lossy computation and an incrementing computation. However, the computational power of faulty approximations far exceeds that of either flavour of unreliable computation in isolation. Indeed, the following theorem demonstrates the equivalence of faulty approximations and reliable computations. Hence, we have what appears to be the first instance of unreliable computation that is, nonetheless, Turing-complete [57].

Theorem 10.6 (Hampson-Kikot-Kurucz [57])**.** *Let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine.*

(i) *If $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L\rangle$ is a reliable computation, then there is some faulty approximation $\langle (q'_k, v_k^-, v_k^+) \in \mathrm{Apx}_{\mathcal{M}} : k < L\rangle$, such that $q_k = q'_k$, for all $k < L$.*

(ii) *If $\langle (q_k, v_k^-, v_k^+) \in \mathrm{Apx}_{\mathcal{M}} : k < L\rangle$ is a faulty approximation, then there is some reliable computation $\langle (q'_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L\rangle$, such that $q_k = q'_k$, for all $k < L$.*

Proof. (i) Suppose that $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L\rangle$ is a reliable computation of $\mathcal{M}$. We may then define a faulty approximation $\langle (q_k, v_k^-, v_k^+) \in \mathrm{Apx}_{\mathcal{M}} : k < L\rangle$, by taking $v_k^-(i) = v_k(i) = v_k^+(i)$, for all $i < n$. It is straightforward to check that this satisfies the definition of a faulty approximation.

(ii) Suppose that $\langle (q_k, v_k^-, v_k^+) \in \mathrm{Apx}_{\mathcal{M}} : k < L\rangle$ is a faulty approximation of $\mathcal{M}$. We construct, by induction on the length, a sequence of configurations $\langle (q'_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < L\rangle$, such that, for all $k < L$:

   (i) $q'_k = q_k$,
   (ii) $v_k^-(i) \leq v_k(i) \leq v_k^+(i)$, for all $i < n$.

   First, by definition, $q_0 = q_{\mathsf{init}}$ and $v_0^+ = v_0^- = \vec{0}$, and so we are free to take $(q_{\mathsf{init}}, \vec{0})$ as our initial configuration.

   Now suppose that $(q'_k, v_k) \in \mathrm{Conf}_{\mathcal{M}}$ has already been defined, for some $k < L - 1$. By the induction hypothesis, we have that $q'_k = q_k$ and $v_k^-(i) \leq v_k(i) \leq v_k^+(i)$, for all $i < n$. Moreover, there is $\alpha \in Op_n$ such that $(q_k, \alpha, q_{k+1}) \in \Delta$.

   We define $v_{k+1} : n \to \omega$, by taking

$$v_{k+1}(i) = \begin{cases} v_k(i) + 1 & \text{if } \alpha = i^{++}, \\ v_k(i) - 1 & \text{if } \alpha = i^{--}, \\ v_k(i) & \text{otherwise,} \end{cases}$$

for all $i < n$.

It remains to show that $(q_k, v_k) \xrightarrow{\mathcal{M}} (q_{k+1}, v_{k+1})$. So suppose that $i < n$, and consider each of the four following cases:

- If $\alpha = i^{++}$ then it follows from the induction hypothesis that
$$v^-_{k+1}(i) \;\leq\; v^-_k(i) + 1 \;\leq\; v_{k+1}(i) \;\leq\; v^+_k(i) + 1 \;\leq\; v^+_{k+1}(i),$$
since $v_{k+1}(i) = v_k(i) + 1$.
- If $\alpha = i^{--}$ then it follows from the induction hypothesis that
$$0 \;\leq\; v^-_{k+1}(i) \;\leq\; v^-_k(i) - 1 \;\leq\; v_{k+1}(i) \;\leq\; v^+_k(i) - 1 \;\leq\; v^+_{k+1}(i),$$
since $v_{k+1}(i) = v_k(i) - 1$.
- If $\alpha = i^{??}$ then it follows from the induction hypothesis that
$$v_k(i) \;\leq\; v^+_k(i) = 0,$$
and hence $v_{k+1}(i) = v_k(i) = 0$.
- In all other cases where $\alpha \in \{j^{++}, j^{--}, j^{??}\}$, for some $j \neq i$, we have that
$$0 \;\leq\; v^-_{k+1}(i) \;\leq\; v^-_k(i) \;\leq\; v_{k+1}(i) \;\leq\; v^+_k(i) \;\leq\; v^+_{k+1}(i),$$
since $v_{k+1}(i) = v_k(i)$.

In all cases we have that $(q_k, v_k) \xrightarrow{\mathcal{M}} (q_{k+1}, v_{k+1})$ and $v^-_{k+1}(i) \leq v_{k+1}(i) \leq v^+_{k+1}(i)$, for all $i < n$. Hence, by induction on the length of the sequence, we can construct an appropriate reliable computation for $\mathcal{M}$.

$\square$

With Theorem 10.6, we find that reachability by faulty approximation is equivalent to reachability by reliable computations. However, the benefit of introducing the notion of faulty approximations is that, as we have seen in Chapter 9, it often requires far less structure to encode unreliable computations than reliable computations.

This is true of our delta products, in which we may express the uniqueness of certain points occurring along the diagonal, but unlike with products of the form $L \times \mathbf{Diff}$, we may not be able to express the uniqueness of points occurring elsewhere. In this chapter we show that these deficiencies are unproblematic for the encoding of faulty approximations.

10.4 Undecidable Delta Products

In Chapter 8, we demonstrated how it is possible to emulate reliable counter machines over product frames, thereby reducing problems of reachability and termination to decision problems for product logics of the form $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{Diff})$. The weak-Euclideanness of frames for **Diff** allow us to express the uniqueness of certain points from among their vertical successors. This expressivity is lacking from **S5**, resulting in considerably more tractable decision problem for corresponding logics of the form $\mathsf{Log}(\mathcal{C} \times \mathsf{Fr}\,\mathbf{S5})$.

In this section we demonstrate, with the aid of faulty approximations, that the presence of a *single* vertically unique element — such as that provided by a diagonal element, in the presence of the axioms $(vdiag_{k,\ell})$, for $k, \ell < \omega$ — is often sufficient for demonstrating undecidability.

10.4.1 Arbitrary Frames

It is well-known that the decision problem for $\mathbf{K} \times \mathbf{S5}$ is coNExpTime-complete [83], and that even the decision problem for $\mathbf{K} \times \mathbf{Diff}$ is decidable, as discussed in Section 6.3. Indeed, both $\mathbf{K} \times \mathbf{S5}$ and $\mathbf{K} \times \mathbf{Diff}$ are known to possess the fmp [83, 118].

However, while the addition of a diagonal element to $\mathbf{S5} \times \mathbf{S5}$ — akin to the addition of equality in the two variable fragment of first-order logic — has no discernible effect of the complexity or size of its models, the same cannot be said of the logics $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{K4}$ and $\mathbf{K} \times \mathbf{S5}$, whose corresponding delta products each lack even in the abstract fmp [74].

As discussed in Section 10.2, the undecidability of the decision problems for both $\mathbf{K} \times^\delta \mathbf{K}$ and $\mathbf{K} \times^\delta \mathbf{K4}$ follows from the undecidability of the global consequence problems for their delta-free counterparts. However, the techniques described there do not extend to $\mathbf{K} \times^\delta \mathbf{S5}$, since the frames for **S5** are not closed under 'spy-points'. Moreover, the global consequence problem for $\mathbf{K} \times \mathbf{S5}$ is decidable [38, Theorem 6.58], and hence would be of little help in establishing any undecidable lower bounds.

In this section, we employ a variation on the counter machine reduction introduced in Chapter 8, to show that the decision problem for $\mathbf{K} \times^\delta \mathbf{S5}$ is undecidable, and is thus strictly more complex than the global consequence problem for $\mathbf{K} \times \mathbf{S5}$.

For each $0 < k \leq \omega$, say that a frame $\mathfrak{F} = (k, R)$ is a *k-fan* if

$$\{(0, n) : 0 < n < k\} \subseteq R.$$

And let $\mathfrak{F}_\omega = (\omega, S)$ denote the ω-fan given by:

$$S = \{(0, n) : 0 < n < \omega\} \cup \{(n, n+1) : 0 < n < \omega\}.$$

With this, we prove the following general theorem.

Theorem 10.7 (Hampson-Kikot-Kurucz [57]). *Let $\mathcal{C}_h$ be any class of frames such that $\mathfrak{F}_\omega \in \mathcal{C}_h$ and let $\mathcal{C}_v$ be any class of frames containing any ω-fan. Then the decision problem for $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$ is Σ^0_1-hard.*

As in Chapter 8, we first fix $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ to be an arbitrary counter machine, and let $\mathfrak{M} = (\mathfrak{F}_h \times^\delta \mathfrak{F}_v, \mathfrak{V})$ be a delta product model, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}_h$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathcal{C}_v$. We construct an $\mathcal{ML}^\delta_2$ formula $\psi_\mathcal{M}$ such that $\psi_\mathcal{M}$ is $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$-satisfiable if and only if $\mathcal{M}$ has a non-terminating *reliable* computation, which we prove by appealing to Theorem 10.6.

To this end, let grid^δ be the conjunction of the following formulas:

$$\Box_v^+ \Diamond_h \delta, \tag{10.5}$$

$$\Box_h \Diamond_v (\Diamond_h \delta \wedge \Box_h \delta). \tag{10.6}$$

Like its precursor grid^{fw}, the purpose of grid^δ is to impose the existence of an infinite grid upon its models, as illustrated in Figure 10.2.

Lemma 10.8. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^\delta$. Then there is some infinite sequence $\langle x_k \in W_h \cap W_v : k < \omega \rangle$ such that, for all $k < \omega$:*

(i) $r_h R_h x_k$,

(ii) $x_0 = r_v$ *and if* $k > 0$ *then* $r_v R_v x_k$,

(iii) *If* $k > 0$ *then* $x_{k-1} R_h x_k$,

(iv) *If* $k > 0$ *then* x_k *is the only* R_h*-successor of* x_{k-1}.

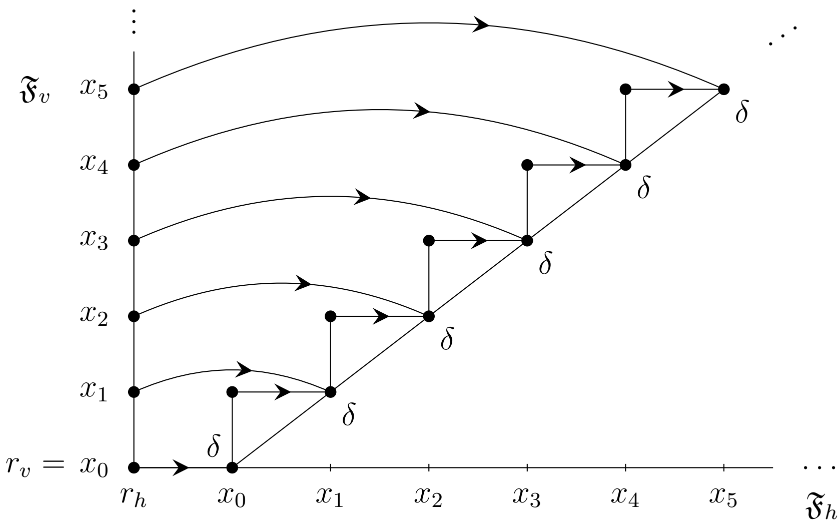


Figure 10.2: Illustration of the grid generated by grid^δ.

Proof. First let $x_0 = r_v$. Then (i) follows from (10.5), while (ii)-(iv) holds vacuously.

Now suppose we have already defined x_k, for some $k < \omega$, satisfying (i)–(iv). It follows that we have $r_h R_h x_k$ and so, by (10.6), there is some $x_{k+1} \in W_v$, such that $r_v R_v x_{k+1}$ and $\mathfrak{M}, (x_k, x_{k+1}) \models \Diamond_h \delta \wedge \Box_h \delta$. Hence, there is some $x \in W_h$, such that $x_k R_h x$ and $\mathfrak{M}, (x, x_{k+1}) \models \delta$, which is to say that $x = x_{k+1}$. Moreover, it follows that x_{k+1} is the only R_h-successor of x_k since $\mathfrak{M}, (x_k, x_{k+1}) \models \Box_h \delta$. $\Box$

With each counter $i < n$, we associate *two* propositional variables p_i^+ and p_i^-, and define:

$$\Sigma_i^+(x) \ := \ \{y \in W_v : r_v R_v y \text{ and } \mathfrak{M}, (x, y) \models p_i^+\},$$

$$\Sigma_i^-(x) \ := \ \{y \in W_v : r_v R_v y \text{ and } \mathfrak{M}, (x, y) \models p_i^-\},$$

for all $x \in W_h$. The intention here is that the value of each counter $i < n$, held at instance $x \in W_h$ is approximated above by the cardinality of $\Sigma_i^+(x)$, and approximated below by the cardinality of $\Sigma_i^-(x)$.

Unlike in the cases considered in Chapter 8, we may only express uniqueness at points along the diagonal. As such, our reliable counting mechanisms described thus far are of little use to us. However, by splitting the computation into two parts — an upper approximation and a lower approximation — we can separate out the unreliability. The strategy will be to allow the upper approximation to incur only *incrementing* errors, while the lower approximation may only incur *lossy* errors, thereby keeping the intended value of the counter bounded, above and below, at all times.

More specifically, for each $i < n$, we define the following formulas:

$$\mathsf{fix}_i^\delta \;:=\; \Box_v^+(p_i^+ \to \Box_h p_i^+) \wedge \Box_v^+(\Diamond_h p_i^- \to p_i^-), \tag{10.7}$$

$$\mathsf{inc}_i^\delta \;:=\; \Box_v^+\big(p_i^+ \to \Box_h p_i^+\big) \wedge \Diamond_v^+(\neg p_i^+ \wedge \Box_h p_i^+) \wedge \Box_v^+\big(\Diamond_h p_i^- \to (p_i^- \vee \delta)\big), \tag{10.8}$$

$$\mathsf{dec}_i^\delta \;:=\; \Box_v\big(p_i^+ \to (\Box_h p_i^+ \vee \delta)\big) \wedge \Box_v^+(\Diamond_h p_i^- \to p_i^-) \wedge \Diamond_v^+(p_i^- \wedge \Box_h \neg p_i^-). \tag{10.9}$$

The following lemma provides the intuition behind these formulas.

Lemma 10.9 (Faulty Counting Lemma)**.** *Suppose that $\mathfrak{M}, (r_h, r_v), \models \mathsf{grid}^\delta$. Then for all $k < \omega$:*

(i) *If $\mathfrak{M}, (x_k, r_v) \models \mathsf{fix}_i^\delta$, then $\Sigma_i^+(x_{k+1}) \supseteq \Sigma_i^+(x_k)$,*

(ii) *If $\mathfrak{M}, (x_k, r_v) \models \mathsf{inc}_i^\delta$, then $\Sigma_i^+(x_{k+1}) \supseteq \Sigma_i^+(x_k) \cup \{z\}$, for some $z \notin \Sigma_i^+(x_k)$,*

(iii) *If $\mathfrak{M}, (x_k, r_v) \models \mathsf{dec}_i^\delta$, then $\Sigma_i^+(x_{k+1}) \supseteq \Sigma_i^+(x_k) - \{x_k\}$,*

(iv) *If $\mathfrak{M}, (x_k, r_v) \models \mathsf{fix}_i^\delta$, then $\Sigma_i^-(x_{k+1}) \subseteq \Sigma_i^-(x_k)$,*

(v) *If $\mathfrak{M}, (x_k, r_v) \models \mathsf{inc}_i^\delta$, then $\Sigma_i^-(x_{k+1}) \subseteq \Sigma_i^-(x_k) \cup \{x_k\}$,*

(vi) *If $\mathfrak{M}, (x_k, r_v) \models \mathsf{dec}_i^\delta$, then $\Sigma_i^-(x_{k+1}) \subseteq \Sigma_i^-(x_k) - \{z\}$, for some $z \in \Sigma_i^-(x_n)$.*

Proof. (i) Suppose that $y \in \Sigma_i^+(x_k)$. Then by definition $r_v R_v y$ and $\mathfrak{M}, (x_k, y) \models p_i^+$. Hence, by (10.7), we have that $\mathfrak{M}, (x_k, y) \models \Box_h p_i^+$, and so $\mathfrak{M}, (x_{k+1}, y) \models p_i^+$, since $x_k R_h x_{k+1}$. This is to say that $y \in \Sigma_i^+(x_{k+1})$.

(ii) By (10.8) there is some $z \in W_h$ such that $r_v R_v^+ z$ and $\mathfrak{M}, (x_k, z) \models \neg p_i^+ \wedge \Box_h p_i^+$. In particular, we have that $z \notin \Sigma_i^+(x_k)$. Moreover, since $x_k R_h x_{k+1}$, we have that $\mathfrak{M}, (x_{k+1}, z) \models p_i$, which is to say that $z \in \Sigma_i^+(x_{k+1})$.

Furthermore, suppose that $y \in \Sigma_i^+(x_k)$. Then by definition $r_v R_v y$ and $\mathfrak{M}, (x_k, y) \models p_i^+$. Hence, by (10.8), we have that $\mathfrak{M}, (x_k, y) \models \Box_h p_i^+$, and so $\mathfrak{M}, (x_{k+1}, y) \models p_i^+$, since $x_k R_h x_{k+1}$. This is to say that $y \in \Sigma_i^+(x_{k+1})$.

(iii) Suppose that $y \in \Sigma_i^+(x_k)$ and $y \neq x_k$. Then by definition $r_v R_v y$ and $\mathfrak{M}, (x_k, y) \models p_i^+$. Hence, by (10.9), we have that $\mathfrak{M}, (x_k, y) \models \Box_h p_i^+ \vee \delta$. However, since $y \neq x_k$, we must have that $\mathfrak{M}, (x_k, y) \models \Box_h p_i^+$, as so it follows that $\mathfrak{M}, (x_{k+1}, y) \models p_i^+$, since $x_k R_h x_{k+1}$. This is to say that $y \in \Sigma_i^+(x_{k+1})$.

(iv) Suppose that $y \in \Sigma_i^-(x_{k+1})$. Then by definition $r_v R_v y$ and $\mathfrak{M}, (x_{k+1}, y) \models p_i^-$. Hence, we have that $\mathfrak{M}, (x_k, y) \models \Diamond_h p_i^-$, since $x_k R_h x_{k+1}$. It then follows from (10.7) that $\mathfrak{M}, (x_k, y) \models p_i^-$, which is to say that $y \in \Sigma_i^-(x_k)$.

(v) Suppose that $y \in \Sigma_i^-(x_{k+1})$. Then by definition $r_v R_v y$ and $\mathfrak{M}, (x_{k+1}, y) \models p_i^-$. Hence, we have that $\mathfrak{M}, (x_k, y) \models \Diamond_h p_i^-$, since $x_k R_h x_{k+1}$. It then follows from (10.8) that $\mathfrak{M}, (x_k, y) \models p_i^- \vee \delta$, which is to say that either $y \in \Sigma_i^-(x_k)$ or $y = x_k$.

(vi) By (10.9) there is some $z \in W_h$ such that $r_v R_v^+ z$ and $\mathfrak{M}, (x_k, z) \models p_i^- \wedge \Box_h \neg p_i^-$. In particular, we have that $z \in \Sigma_i^-(x_k)$. Moreover, since $x_k R_h x_{k+1}$, we have that $\mathfrak{M}, (x_{k+1}, z) \models \neg p_i^-$, which is to say that $z \notin \Sigma_i^-(x_{k+1})$.

Now suppose that $y \in \Sigma_i^-(x_{k+1})$. Then by definition $r_v R_v y$ and $\mathfrak{M}, (x_{k+1}, y) \models p_i^-$. Hence, we have that $\mathfrak{M}, (x_k, y) \models \Diamond_h p_i^-$, since $x_k R_h x_{k+1}$. It then follows from (10.9) that $\mathfrak{M}, (x_k, y) \models p_i^-$, which is to say that $y \in \Sigma_i^-(x_k)$. Moreover, $y \neq z$ since $z \notin \Sigma_i^-(x_{k+1})$.

$\Box$

We specify the action of each counter operation $\alpha \in Op_n$ by the following combination of these atomic actions on each of the individual counter variables:

$$\mathsf{Do}_\alpha^\delta \;:=\; \begin{cases} \mathsf{inc}_i^\delta \wedge \bigwedge_{i \neq j} \mathsf{fix}_j^\delta & \text{if } \alpha = i^{++}, \\ \mathsf{dec}_i^\delta \wedge \bigwedge_{i \neq j} \mathsf{fix}_j^\delta & \text{if } \alpha = i^{--}, \\ \Box_v^+ \neg p_i^+ \wedge \bigwedge_{j<n} \mathsf{fix}_j^\delta & \text{if } \alpha = i^{??}. \end{cases}$$

Finally, for each state $q \in Q$, let S_q be a fresh propositional variable and take $\varphi^\delta_{\mathcal{M}}$ to be the conjunction of the following formulas:

$$\Box_h\Big(\delta \to \big(\widehat{S}_{q_{\mathsf{init}}} \ \wedge \ \bigwedge_{i<n} \Box^+_v(\neg p^+_i \wedge \neg p^-_i)\big)\Big), \tag{10.10}$$

$$\Box_h \bigwedge_{q \in Q-H} \Big(\widehat{S}_q \ \to \bigvee_{(q,\alpha,q') \in \Delta} \big(\Box_h \widehat{S}_{q'} \wedge \mathsf{Do}^\delta_\alpha\big)\Big), \tag{10.11}$$

$$\Box_h \bigwedge_{h \in H} \neg \widehat{S}_h, \tag{10.12}$$

where $\widehat{S}_q := S_q \wedge \bigwedge_{q' \neq q} \neg S_{q'}$.

The first conjunct specifies the initial starting configuration $(q_0, \vec{0}, \vec{0}) \in \mathrm{Apx}_{\mathcal{M}}$, while the second governs the behaviour of the machine in accordance to the instructions of $\mathcal{M}$. The last conjunct stipulates that the approximation be non-terminating. These details are more formally addressed by the following lemma.

Lemma 10.10 (Faulty Emulation Lemma). *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^\delta \wedge \varphi^\delta_{\mathcal{M}}$ and let $\langle x_k \in W_h \cap W_v : k < \omega\rangle$ be any infinite sequence satisfying conditions (i)–(iv) of Lemma 10.13. Then $\mathcal{M}$ has a non-terminating faulty approximation $\langle (q_k, v^+_k, v^-_k) : k < \omega\rangle$ such that $\mathfrak{M}, (x_k, r_v) \models \widehat{S}_{q_k}$, for all $k < \omega$.*

Proof. We construct, by induction on the length, an infinite sequence of approximants $\langle (q_k, v^+_k, v^-_k) \in \mathrm{Apx}_{\mathcal{M}} : k < \omega\rangle$ such that $q_0 = q_{\mathsf{init}}$, $v^+_0 = v^-_0 = \vec{0}$, and for all $k < \omega$:

(i) $\mathfrak{M}, (x_k, r_v) \models \widehat{S}_{q_k}$,

(ii) $v^+_k(i) \leq |\Sigma^+_i(x_k)|$, for all $i < n$,

(iii) $v^-_k(i) = |\Sigma^-_i(x_k)|$, for all $i < n$,

(iv) If $k > 0$ then $(q_{k-1}, v^+_{k-1}, v^-_{k-1}) \xrightarrow{\mathcal{M}^{\approx}} (q_k, v^+_k, v^-_k)$.

First, by Lemma 10.13, we have that $r_h R_h x_0$, $x_0 = r_v$, and so it follows from (10.10) that $\mathfrak{M}, (x_0, r_v) \models \widehat{S}_{q_{\mathsf{init}}}$ and $\mathfrak{M}, (x_0, r_v) \models \Box^+_v(\neg p^+_i \wedge \neg p^-_i)$, for all $i < n$. We may then take, as our first approximant, the tuple (q_0, v^+_0, v^-_0) where $q_0 = q_{\mathsf{init}}$ and $v^+_0 = v^-_0 = \vec{0}$.

Now suppose that (q_k, v_k^+, v_k^-) has already been defined, for some $k < \omega$. By the induction hypothesis, we have that $\mathfrak{M}, (x_k, r_v) \models \widehat{S}_{q_k}$, while it follows from (10.12) that $q_k \notin H$. Thus we may infer from (10.11) that

$$\mathfrak{M}, (x_{k+1}, r_v) \models \Box_h \widehat{S}_{q_{k+1}} \wedge \mathsf{Do}_\alpha,$$

for some $(q_k, \alpha, q_{k+1}) \in \Delta$, since $r_h R_h x_k$. It then follows that $\mathfrak{M}, (x_{k+1}, r_v) \models \widehat{S}_{q_{k+1}}$, since $x_k R_h x_{k+1}$, thereby satisfying (i).

We define $v_{k+1}^+, v_{k+1}^- : n \to \omega$, by taking

$$v_{k+1}^+(i) = \begin{cases} |\Sigma_i^+(x_{k+1})| & \text{if } |\Sigma_i^+(x_{k+1})| \text{ is finite,} \\ v_k^+(i) + 1 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad v_{k+1}^-(i) = |\Sigma_i^-(x_{k+1})|,$$

for all $i < n$, thereby satisfying (ii) and (iii).

It remains to show that $(q_k, v_k^+, v_k^-) \stackrel{\mathcal{M}^{\approx}}{\longrightarrow} (q_{k+1}, v_{k+1}^+, v_{k+1}^-)$. So suppose that $i < n$, and consider each of the four following cases, each of which follows from Lemma 10.9 and the induction hypothesis:

- If $\alpha = i^{++}$, then $\mathfrak{M}, (x_k, r_v) \models \mathsf{inc}_i^\delta$. Firstly, if $|\Sigma_i^+(x_{k+1})|$ is infinite then it follows immediately from the definition that $v_{k+1}^+(i) \geq v_k^+(i) + 1$. Otherwise, we have that

$$\begin{aligned} v_{k+1}^+(i) = |\Sigma_i^+(x_{k+1})| &\geq |\Sigma_i^+(x_k)| + 1 \geq v_k^+(i) + 1, \\ v_{k+1}^-(i) = |\Sigma_i^-(x_{k+1})| &\leq |\Sigma_i^-(x_k)| + 1 = v_k^-(i) + 1. \end{aligned}$$

- If $\alpha = i^{--}$, then $\mathfrak{M}, (x_k, r_v) \models \mathsf{dec}_i^\delta$. If $|\Sigma_i^+(x_{k+1})|$ is infinite then it follows immediately from the definition that $v_{k+1}^+(i) = v_k^+(i) + 1 \geq v_k^+(i) - 1$. Otherwise, we have that

$$\begin{aligned} v_{k+1}^+(i) = |\Sigma_i^+(x_{k+1})| &\geq |\Sigma_i^+(x_k)| - 1 \geq v_k^+(i) - 1, \\ v_{k+1}^-(i) = |\Sigma_i^-(x_{k+1})| &\leq |\Sigma_i^-(x_k)| - 1 = v_k^-(i) - 1. \end{aligned}$$

- If $\alpha = i^{??}$, then $\mathfrak{M}, (x_k, r_v) \models \Box_v^+ \neg p_i^+ \wedge \mathsf{fix}_i^\delta$, and so it follows that

$$v_k^+(i) \leq |\Sigma_i^+(x_k)| = 0.$$

Moreover, if $|\Sigma_i^+(x_{k+1})|$ is infinite then $v_{k+1}^+(i) = v_k^+(i) + 1 \geq v_k^+(i)$. Otherwise, it follows that

$$\begin{aligned} v_{k+1}^+(i) = |\Sigma_i^+(x_{k+1})| &\geq |\Sigma_i^+(x_k)| \geq v_k^+(i), \\ v_{k+1}^-(i) = |\Sigma_i^-(x_{k+1})| &\leq |\Sigma_i^-(x_k)| = v_k^-(i). \end{aligned}$$

– In all other cases where $\alpha \in \{j^{++}, j^{--}, j^{??}\}$, for $j \neq i$, we find that $\mathfrak{M}, (x_k, r_v) \models \mathsf{fix}_i^\delta$. If $|\Sigma_i^+(x_{k+1})|$ is infinite then $v_{k+1}^+(i) = v_k^+(i) + 1 \geq v_k^+(i)$. Otherwise, it then follows that

$$\begin{aligned} v_{k+1}^+(i) = |\Sigma_i^+(x_{k+1})| &\geq |\Sigma_i^+(x_k)| \geq v_k^+(i), \\ v_{k+1}^-(i) = |\Sigma_i^-(x_{k+1})| &\leq |\Sigma_i^-(x_k)| = v_k^-(i). \end{aligned}$$

Thus, we conclude that $(q_k, v_k^+, v_k^-) \xrightarrow{\mathcal{M}^{\approx}} (q_{k+1}, v_{k+1}^+, v_{k+1}^-)$, thereby satisfying (iv). Hence, by induction on the length of the sequence, we can construct an appropriate non-terminating faulty approximation for $\mathcal{M}$, as required. □

We are now in a position to prove Theorem 10.7.

Proof of Theorem 10.7. Let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, and take

$$\psi_{\mathcal{M}} = \mathsf{grid}^\delta \wedge \varphi_{\mathcal{M}}^\delta.$$

We prove that the following statements are equivalent:

(i) $\mathcal{M}$ has a non-terminating faulty approximation,

(ii) $\mathcal{M}$ has a non-terminating reliable computation,

(iii) $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$-satisfiable.

(i) ⇒ (ii) This follows immediately from Theorem 10.6.

(ii) ⇒ (iii) Suppose that $\mathcal{M}$ has a non-terminating reliable computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$. We define the model $\mathfrak{M} = (\mathfrak{F}_h \times^\delta \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = \mathfrak{F}_\omega = (\omega, S) \in \mathcal{C}_h$ is the ω-fan defined above, and $\mathfrak{F}_v = (\omega, R_v) \in \mathcal{C}_v$ is any arbitrary ω-fan, by taking

$$\mathfrak{V}(S_q) = \{(k, 0) : k < \omega \text{ and } q = q_k\}, \qquad \text{for each } q \in Q.$$

For each $i < n$, we define the sets $\mu_k(p_i^+)$ and $\mu_k(p_i^-)$, for $k < \omega$, and take

$$\mathfrak{V}(p_i^+) = \{(k,m) : m \in \mu_k(p_i^+)\} \qquad \text{and} \qquad \mathfrak{V}(p_i^-) = \{(k,m) : m \in \mu_k(p_i^-)\}.$$

First, let $\mu_0(p_i^+) = \mu_0(p_i^-) = \emptyset$ and define

$$\mu_{k+1}(p_i^-) = \begin{cases} \mu_k(p_i^-) \cup \{k\} & \text{if } v_{k+1}(i) > v_k(i), \\ \mu_k(p_i^-) - \{\min \mu_k(p_i^-)\} & \text{if } v_{k+1}(i) < v_k(i), \\ \mu_k(p_i^-) & \text{otherwise,} \end{cases}$$

for all $k < \omega$.

We define $\mu_k(p_i^-)$ similarly, for $k < \omega$. However, here, we must take a more imaginative approach, as we are only permitted to remove elements along the diagonal. For this reason we must 'anticipate' the order in which the decrements are to occur, and choose those increments which mirror this ordering.

For example, if our first decrement is to occur in the third instance, then our first increment should be placed in the third row, so that it crosses the diagonal at precisely the time it is to be decremented.

To this end, for each $i < n$, let

$$\Lambda(i) = \{k < \omega : v_{k+1}(i) > v_k(i)\} \quad \text{and} \quad \Xi(i) = \{k < \omega : v_{k+1}(i) < v_k(i)\}$$

be those instances where counter $i < n$ is incremented and decremented, respectively. Let $\lambda_i : \Lambda(i) \to \omega$ be such that

$$\lambda_i(k) = \begin{cases} \min(\Xi_k(i)) & \text{if } \Xi_k(i) \neq \emptyset, \\ k+1 & \text{otherwise,} \end{cases} \tag{10.13}$$

where $\Xi_n(i) = \{k < \omega : k \geq n \text{ and } v_{k+1}(i) < v_k(i)\} \subseteq \Xi(i)$. Note that, by definition, $\lambda_i(k) \geq k$, for all $k \in \Lambda(i)$, while a straightforward induction reveals that $\Xi(i) \subseteq \mathsf{Rng}(\lambda_i)$, where $\mathsf{Rng}(\lambda_i)$ is the range of λ_i.

Intuitively, λ_i maps each instant where counter i is incremented, with a later instant where counter i is decremented — should such a decrement occur. In the case where no further decrements occur to counter i, the function simply returns the next instant of time, although any later time would serve the same purpose.

We then define

$$\mu_{k+1}(p_i^+) = \begin{cases} \mu_k(p_i^+) \cup \{\lambda_i(k)\} & \text{if } v_{k+1}(i) > v_k(i), \\ \mu_k(p_i^+) - \{k\} & \text{if } v_{k+1}(i) < v_k(i), \\ \mu_k(p_i^+) & \text{otherwise.} \end{cases}$$

It is then straightforward to check that $\mathfrak{M}, (\omega, 0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$-satisfiable.

(iii) $\Rightarrow$ (i) Lastly, suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$-satisfiable then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$ for some model $\mathfrak{M} = (\mathfrak{F}_h \times^\delta \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}_h$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathcal{C}_v$.

It follows immediately from Lemma 10.10, that $\mathcal{M}$ has a non-terminating faulty approximation.

Now, since the Termination problem for reliable counter machines is Σ_1^0-hard, so too must be the decision problem for $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$, as required. □

It now follows from Theorem 10.7 that the decision problem for $\mathbf{K} \times^\delta \mathbf{S5}$ is undecidable; taking $\mathcal{C}_h$ to be the class of all Kripke frames and $\mathcal{C}_v$ to be the class of all equivalence relations. Moreover, since both $\mathbf{K}$ and $\mathbf{S5}$ are characterised by some first-order definable class of frames, it follows from a straightforward generalisation of Theorem 3.5 that $\mathbf{K} \times^\delta \mathbf{S5}$ recursively enumerable.

Corollary 10.11. *The decision problem for* $\mathbf{K} \times^\delta \mathbf{S5}$ *is* Σ_1^0*-complete.*

Furthermore, Theorem 10.7 also provides alternative proofs for the undecidability of the decision problems for $\mathbf{K} \times^\delta \mathbf{K}$ and $\mathbf{K} \times^\delta \mathbf{K4}$, whose undecidability also follows from the undecidability of the global consequence problems for $\mathbf{K} \times \mathbf{K}$ and $\mathbf{K} \times \mathbf{K4}$, respectively (see Section 10.2).

10.4.2 Linear Frames

In this section, we combine the new ideas of the previous section with those of Section 8.3.1, to prove that, unlike **K4.3**$\times$**K** and **K4.3**$\times$**S5**, whose decision problems are both decidable — indeed, even in 2ExpTime for **K4.3** $\times$ **S5** [101, 38] — the decision problems for their respective delta products are undecidable.

Theorem 10.12 (Hampson-Kikot-Kurucz [57]). *Let $\mathcal{C}_h$ be any class of strict linear orders such that $(\omega, <) \in \mathcal{C}_h$, and let $\mathcal{C}_v$ be any class of frames containing an ω-fan. Then the decision problem for $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$ is Σ^0_1-hard.*

As always, let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, and let $\mathfrak{M} = (\mathfrak{F}_h \times^\delta \mathfrak{F}_v, \mathfrak{V})$ be a delta product model, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}_h$ is a strict linear order and $\mathfrak{F}_v = (W_v, R_v) \in \mathcal{C}_v$.

First, we define grid^{fw} as in the proof of Theorem 8.8, for which we have the following lemma, repeated here for reference.

Lemma 10.13. *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw}$. Then there are two infinite sequences $\langle a_k \in W_h : k < \omega\rangle$ and $\langle b_k \in W_v : k < \omega\rangle$, such that, for all $k < \omega$:*

(i) *$a_0 = r_h$ and if $k > 0$ then a_k is the immediate successor of a_{k-1},*

(ii) *$b_0 = r_v$ and if $k > 0$ then $r_v R_v b_k$,*

(iii) *$\mathfrak{M}, (a_k, b_k) \models c$,*

(iv) *If $k > 0$ then $\mathfrak{M}, (a_{k-1}, b_k) \models d$.*

Proof. See Lemma 8.9. □

Here, we diverge from the proof of Theorem 8.8; introducing instead *four* propositional variables p_i^+, q_i^+, p_i^- and q_i^-, for each $i < n$, together with the auxiliary variables $\mathsf{start}(p)$, for all $p \in \{p_i^+, p_i^-, q_i^+, q_i^- : i < n\}$. For each $i < n$, we define

$$\Sigma_i^+(x) := \{y \in W_v : r_v R_v^+ y \text{ and } \mathfrak{M}, (x, y) \models p_i^+ \wedge \neg q_i^+\},$$
$$\Sigma_i^-(x) := \{y \in W_v : r_v R_v^+ y \text{ and } \mathfrak{M}, (x, y) \models p_i^- \wedge \neg q_i^-\},$$

for all $x \in W_h$. As stated above, the intention here is to approximate the value of each counter $i < n$, held at instance $x \in W_h$, by bounding it above and below by the cardinalities of $\Sigma_i^+(x)$ and $\Sigma_i^-(x)$, respectively.

For each $p \in \{p_i^+, q_i^+, p_i^-, q_i^- : i < n\}$, we define $\mathsf{counter}(p)$ as in Section 8.3.1, and take $\mathsf{counter}^\delta$ to be their combined conjunction together with the formula

$$\bigwedge_{i<n} \Box_h^+ \Box_v^+ (q_i^+ \to p_i^+) \wedge \bigwedge_{i<n} \Box_h^+ \Box_v^+ (q_i^- \to p_i^-), \tag{10.14}$$

stipulating that we cannot mark any counter as 'off' until it has first been marked as 'on'.

For each $i < n$, we define the following formulas:

$$\mathsf{fix}_i^\delta \;:=\; \Box_v^+ \neg\mathsf{start}(q_i^+) \wedge \Box_v^+ \neg\mathsf{start}(p_i^-), \tag{10.15}$$

$$\mathsf{inc}_i^\delta \;:=\; \Diamond_v^+ \mathsf{start}(p_i^+) \wedge \Box_v^+ \neg\mathsf{start}(q_i^+) \wedge \Box_v^+ (\mathsf{start}(p_i^-) \to \delta), \tag{10.16}$$

$$\mathsf{dec}_i^\delta \;:=\; \Box_v^+ (\mathsf{start}(q_i^+) \to \delta) \wedge \Diamond_v^+ \mathsf{start}(q_i^-) \wedge \Box_v^+ \neg\mathsf{start}(p_i^-), \tag{10.17}$$

whose interpretation is explained by way of the following lemma.

Lemma 10.14 (Counting Lemma)**.** *Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw} \wedge \mathsf{counter}^\delta$. Then for all $k < \omega$:*

(i) *If $\mathfrak{M}, (a_k, r_v) \models \mathsf{fix}_i^\delta$, then $\Sigma_i^+(a_{k+1}) \supseteq \Sigma_i^+(a_k)$,*

(ii) *If $\mathfrak{M}, (a_k, r_v) \models \mathsf{inc}_i^\delta$, then $\Sigma_i^+(a_{k+1}) \supseteq \Sigma_i^+(a_k) \cup \{z\}$, for some $z \notin \Sigma_i^+(a_k)$,*

(iii) *If $\mathfrak{M}, (a_k, r_v) \models \mathsf{dec}_i^\delta$, then $\Sigma_i^+(a_{k+1}) \supseteq \Sigma_i^+(a_k) - \{a_k\}$,*

(iv) *If $\mathfrak{M}, (a_k, r_v) \models \mathsf{fix}_i^\delta$, then $\Sigma_i^-(a_{k+1}) \subseteq \Sigma_i^-(a_k)$,*

(v) *If $\mathfrak{M}, (a_k, r_v) \models \mathsf{inc}_i^\delta$, then $\Sigma_i^-(a_{k+1}) \subseteq \Sigma_i^-(a_k) \cup \{a_k\}$,*

(vi) *If $\mathfrak{M}, (a_k, r_v) \models \mathsf{dec}_i^\delta$, then $\Sigma_i^-(a_{k+1}) \subseteq \Sigma_i^-(a_k) - \{z\}$, for some $z \in \Sigma_i^-(a_k)$.*

Proof. (i) Suppose $y \in \Sigma_i^+(a_k)$ then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_k, y) \models p_i^+ \wedge \neg q_i^+$. By (10.15), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(q_i^+)$. It then follows from Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i^+ \wedge \neg q_i^+$. This is to say that $y \in \Sigma_i^+(a_{k+1})$, as required.

(ii) By (10.16), there is some $z \in W_v$ such that $r_h R_h^+ z$ and $\mathfrak{M}, (a_k, z) \models \mathsf{start}(p_i^+) \wedge \neg\mathsf{start}(q_i^+)$ (however, this z need not be unique). By Lemma 8.10 we have that $\mathfrak{M}, (a_k, z) \models \neg p_i^+$ and $\mathfrak{M}, (a_{k+1}, z) \models p_i^+$. It follows from (8.12) that $\mathfrak{M}, (a_k, z) \models \neg q_i^+$ and so, by Lemma 8.10, $\mathfrak{M}, (a_{k+1}, z) \models \neg q_i^+$. Hence $z \notin \Sigma_i^+(x_k)$ and $z \in \Sigma_i^+(a_{k+1})$.

Now suppose $y \in \Sigma_i^+(a_k)$ then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_k, y) \models p_i^+ \wedge \neg q_i^+$. By (10.16) we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(q_i^+)$. Again, it then follows from Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i^+ \wedge \neg q_i^+$. This is to say that $y \in \Sigma_i^+(a_{k+1})$.

(iii) Suppose that $y \in \Sigma_i^+(a_k)$ and $y \neq a_k$ then by definition $r_v R_v^+ y$, $\mathfrak{M}, (a_k, y) \models p_i^+ \wedge \neg q_i^+$ and $\mathfrak{M}, (a_k, y) \models \neg\delta$. Hence by (10.17), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(q_i^+)$. It then follows from Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i^+ \wedge \neg q_i^+$. This is to say that $y \in \Sigma_i^+(a_{k+1})$.

(iv) Suppose that $y \in \Sigma_i^-(a_{k+1})$ then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_{k+1}, y) \models p_i^- \wedge \neg q_i^-$. By (10.15), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(p_i^-)$. It then follows from Lemma 8.10 that $\mathfrak{M}, (a_k, y) \models p_i^- \wedge \neg q_i^-$. This is to say that $y \in \Sigma_i^-(a_k)$, as required.

(v) Suppose that $y \in \Sigma_i^-(a_{k+1})$ and $y \neq a_k$. Then by definition $r_v R_v^+ y$, $\mathfrak{M}, (a_{k+1}, y) \models p_i^- \wedge \neg q_i^-$ and $\mathfrak{M}, (a_k, y) \models \neg\delta$. Hence, by (10.16), we have that $\mathfrak{M}, (a_k, y) \models \neg\mathsf{start}(p_i^-)$. It then follows from Lemma 8.10 that $\mathfrak{M}, (a_{k+1}, y) \models p_i^+ \wedge \neg q_i^+$. This is to say that $y \in \Sigma_i^-(a_k)$.

(vi) By (10.17), there is some $z \in W_v$ such that $r_h R_h^+ z$ and $\mathfrak{M}, (a_k, z) \models \mathsf{start}(q_i^-) \wedge \neg\mathsf{start}(p_i^-)$ (however this z need not be unique). By Lemma 8.10 we have that $\mathfrak{M}, (a_k, z) \models \neg q_i^-$ and $\mathfrak{M}, (a_{k+1}, z) \models q_i^-$. It follows from (8.12) that $\mathfrak{M}, (a_{k+1}, z) \models p_i^-$ and so, by Lemma 8.10, $\mathfrak{M}, (a_k, z) \models p_i^-$. Hence $z \in \Sigma_i^-(a_k)$ and $z \notin \Sigma_i^-(a_{k+1})$.

Now suppose $y \in \Sigma_i^+(a_{k+1})$ then by definition $r_v R_v^+ y$ and $\mathfrak{M}, (a_{k+1}, y) \models p_i^- \wedge \neg q_i^-$. By (10.17) we have that $\mathfrak{M}, (a_{k+1}, y) \models \neg\mathsf{start}(p_i^-)$. Again, it then follows from Lemma 8.10 that $\mathfrak{M}, (a_k, y) \models p_i^- \wedge \neg q_i^-$. This is to say that $y \in \Sigma_i^-(a_k)$.

$\square$

We then specify the action of each of the counter operation $\alpha \in Op_n$ as follows:

$$\mathsf{Do}_\alpha^\delta \;:=\; \begin{cases} \mathsf{inc}_i^\delta \wedge \bigwedge_{j \neq i} \mathsf{fix}_j^\delta & \text{if } \alpha = i^{++}, \\ \mathsf{dec}_i^\delta \wedge \bigwedge_{j \neq i} \mathsf{fix}_j^\delta & \text{if } \alpha = i^{--}, \\ \Box_v^+(p_i^+ \to q_i^+) \wedge \bigwedge_{j<n} \mathsf{fix}_j^\delta & \text{if } \alpha = i^{??}. \end{cases}$$

For each state $q \in Q$, let S_q be a fresh propositional variable and take $\varphi^{\delta}_{\mathcal{M}}$ to be the conjunction of the following formulas:

$$\widehat{S}_{q_{\mathsf{init}}} \ \wedge \ \bigwedge_{i<n} \Box_v^+(\neg p_i^+ \wedge \neg p_i^-)\big), \tag{10.18}$$

$$\Box_h^+ \bigwedge_{q \in Q-H} \Big(\Diamond_v^+(c \wedge \widehat{S}_q) \ \rightarrow \bigvee_{(q,\alpha,q') \in \Delta} \big(\mathsf{Do}_\alpha \wedge \Box_v^+\big(d \rightarrow \Box_h(c \rightarrow \widehat{S}_{q'})\big)\big)\Big), \tag{10.19}$$

$$\Box_h^+ \Box_v^+ \bigwedge_{h \in H} \neg \widehat{S}_h, \tag{10.20}$$

where $\widehat{S}_q := S_q \wedge \bigwedge_{q' \neq q} \neg S_{q'}$. It is then straightforward to prove the following emulation lemma, analogous to that of Lemma 8.12.

Lemma 10.15 (Approximation Emulation Lemma)**.** *Suppose that* $\mathfrak{M}, (r_h, r_v) \models \mathsf{grid}^{fw} \wedge \mathsf{counter}^{\delta} \wedge \varphi^{\delta}_{\mathcal{M}}$*. Then* $\mathcal{M}$ *has a non-terminating faulty approximation* $\langle (q_k, v_k^+, v_k^-) : k < \omega \rangle$ *such that* $\mathfrak{M}, (a_k, r_v) \models \widehat{S}_{q_k}$*, for all* $k < \omega$.

Proof. Analogous to that of Lemma 8.12. $\Box$

We are now in a position to prove Theorem 10.12.

Proof of Theorem 10.12. Let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary counter machine, and take

$$\psi_{\mathcal{M}} \ := \ \mathsf{grid}^{fw} \wedge \mathsf{counter}^{\delta} \wedge \varphi^{\delta}_{\mathcal{M}}.$$

We prove that the following statements are equivalent:

(i) $\mathcal{M}$ has a non-terminating faulty approximation,

(ii) $\mathcal{M}$ has a non-terminating reliable computation,

(iii) $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C}_h \times^{\delta} \mathcal{C}_v)$-satisfiable.

(i) $\Rightarrow$ (ii) This follows immediately from Theorem 10.6.

(ii) $\Rightarrow$ (iii) Suppose that $\mathcal{M}$ has a non-terminating reliable computation $\langle (q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}} : k < \omega \rangle$. Let $\mathfrak{F}_h = (\omega, <) \in \mathcal{C}_h$ and let $\mathfrak{F}_v = (\omega, R_v) \in \mathcal{C}_v$ be an ω-fan and define the model $\mathfrak{M} = (\mathfrak{F}_h \times^\delta \mathfrak{F}_v, \mathfrak{V})$, by taking

$$\begin{aligned}
\mathfrak{V}(c) &= \{(k,k) : k < \omega\}, \\
\mathfrak{V}(d) &= \{(k+1,k) : k < \omega\}, \\
\mathfrak{V}(S_q) &= \{(k,k) : k < \omega \text{ and } q = q_k\}, \qquad \text{for each } q \in Q.
\end{aligned}$$

For each $p \in \{p_i^+, q_i^+, p_i^-, q_i^- : i < n\}$, we define the sets $\mu_k(p)$ inductively, for $k < \omega$, and take

$$\mathfrak{V}(p) = \{(k,m) : m \in \mu_k(p)\}.$$

First, let $\mu_0(p) = \emptyset$, for all $p \in \{p_i^+, q_i^+, p_i^-, q_i^- : i < n\}$, and for each $k < \omega$, we define

$$\begin{aligned}
\mu_{k+1}(p_i^-) &= \begin{cases} \mu_k(p_i^-) \cup \{k\} & \text{if } v_{k+1}(i) > v_k(i), \\ \mu_k(p_i^-) & \text{otherwise,} \end{cases} \\
\mu_{k+1}(q_i^-) &= \begin{cases} \mu_k(q_i^-) \cup \{\min\big(\mu_k(p_i^-) - \mu_k(q_i^-)\big)\} & \text{if } v_{k+1}(i) < v_k(i), \\ \mu_k(q_i^-) & \text{otherwise,} \end{cases} \\
\mu_{k+1}(q_i^+) &= \begin{cases} \mu_k(q_i^+) \cup \{k\} & \text{if } v_{k+1}(i) < v_k(i), \\ \mu_k(q_i^+) & \text{otherwise.} \end{cases}
\end{aligned}$$

Lastly, by recalling the notation introduced in (10.13), we take

$$\mu_{k+1}(p_i^+) = \begin{cases} \mu_k(p_i^+) \cup \{\lambda_i(k)\} & \text{if } v_{k+1}(i) > v_k(i), \\ \mu_k(p_i^+) & \text{otherwise.} \end{cases}$$

It follows from a straightforward induction that $\mu_k(q_i^-) \subseteq \mu_k(p_i^-)$, for all $k < \omega$. What is less obvious is that $\mu_k(q_i^+) \subseteq \mu_k(p_i^+)$, for all $k < \omega$.

- By definition we have that $\mu_0(q_i^+) \subseteq \mu_0(p_i^+)$, so suppose that $\mu_k(q_i^+) \subseteq \mu_k(p_i^+)$, for some $k < \omega$, and let $\ell \in \mu_{k+1}(q_i^+)$. We have two cases to consider:
  * If $\ell \in \mu_k(q_i^+)$, then it follows from the induction hypothesis that $\ell \in \mu_k(p_i^+)$, and hence $\ell \in \mu_{k+1}(p_i^-)$, as required.
  * Otherwise, $\ell = k$ and $v_{k+1}(i) < v_k(i)$. It then follows from the above definition

that $k \in \Xi(i)$. Furthermore, as noted above, $\Xi(i) \subseteq \mathsf{Rng}(\lambda_i)$, and so there is some $k' \in \Lambda(i)$ such that $\lambda_i(k') = k$. Since $k' \in \Lambda(i)$, we have that $v_{k'+1}(i) > v_{k'}(i)$, and hence, by definition $k = \lambda_i(k') \in \mu_{k'+1}(p_i^+)$. However, since $k' \leq \lambda_i(k') = k$, we must have that $k \in \mu_{k+1}(p_i^+)$, as required.

It is then straightforward to check that $\mathfrak{M}, (0,0) \models \psi_{\mathcal{M}}$, and hence it follows that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$-satisfiable.

(iii) $\Rightarrow$ (i) Lastly, suppose that $\psi_{\mathcal{M}}$ is $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$-satisfiable then $\mathfrak{M}, (r_h, r_v) \models \psi_{\mathcal{M}}$ for some model $\mathfrak{M} = (\mathfrak{F}_h \times^\delta \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_h = (W_h, R_h) \in \mathcal{C}_h$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathcal{C}_v$.

It then follows immediately from Lemma 10.10 that $\mathcal{M}$ has a non-terminating faulty approximation.

Now, since the Termination problem for reliable counter machines is Σ_1^0-hard, so too must be the decision problem for $\mathsf{Log}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$, as required. □

It then follows from Theorem 10.12 that the decision problem for each of the logics $\mathbf{K4.3} \times^\delta \mathbf{K}$ and $\mathbf{K4.3} \times^\delta \mathbf{S5}$ is undecidable. Moreover, since each of these logics is characterised by some first-order definable class of frames, it follows from Theorem 3.5 that they are each recursively enumerable.

Corollary 10.16. *The decision problems for* $\mathbf{K4.3} \times^\delta \mathbf{K}$ *and* $\mathbf{K4.3} \times^\delta \mathbf{S5}$ *are* Σ_1^0*-complete.*

It should be noted that the undecidability of the decision problem for $\mathbf{K4.3} \times^\delta \mathbf{K4}$ also follows from Theorem 10.12, however it is already well-established that the decision problem for $\mathbf{K4.3} \times \mathbf{K4}$ is undecidable [102], and the addition of a diagonal element does nothing to mitigate this.

It should be noted that the decision problem for $\mathbf{K4.3} \times^\delta \mathbf{S5}$ is strictly more complex than both the decision problem and global consequence problem for $\mathbf{K4.3} \times \mathbf{S5}$, both of which are decidable in 2ExpTime [101, 38].

Furthermore, while the decision problem for $\mathbf{K4.3} \times \mathbf{K}$ is known to be decidable, the decidability of its global consequence problem remains open. However, even if it were shown that the problem for deciding the global consequences of $\mathbf{K4.3} \times \mathbf{K}$ was undecidable, we would still be unable to infer the undecidability of the decision problem for $\mathbf{K4.3} \times^\delta \mathbf{K}$, since the frames for $\mathbf{K4.3}$ are not closed under the addition of ‘spy-points’, required by Proposition 10.3.

10.5 Logics of Bounded Width

The most notable exceptions from the scope of Theorem 10.7 are the logics $\mathbf{K} \times^\delta \mathbf{Alt}(t)$, for $0 < t < \omega$, where $\mathbf{Alt}(t)$ is the logic characterised by the class of all frames $\mathfrak{F} = (W, R)$ whose worlds have no more than t distinct R-successors. This is a generalisation of the logic $\mathbf{Alt} := \mathbf{Alt}(1)$, characterised by those frames whose accessibility relation describes a partial function, discussed in Section 5.3. It is clear that, $\mathbf{Alt}(t)$ admits no ω-fans among its frames, and thus $\mathbf{K} \times^\delta \mathbf{Alt}(t)$ lies outside the remit of Theorem 10.7.

In this section we show that, in contrast with those logics such as $\mathbf{K} \times^\delta \mathbf{K}$, $\mathbf{K} \times^\delta \mathbf{S5}$, and $\mathbf{K} \times^\delta \mathbf{K4}$ whose decision problems are undecidable, each of the logics $\mathbf{K} \times^\delta \mathbf{Alt}(t)$ enjoy the exponential product fmp, for $0 < t < \omega$, and hence their decision problems are each decidable in coNExpTime.

Theorem 10.17. $\mathbf{K} \times^\delta \mathbf{Alt}(t)$ *enjoys the exponential product fmp, for all* $0 < t < \omega$.

Proof. Suppose that $0 < t < \omega$ and let $\varphi \in \mathcal{ML}_2^\delta$ be an arbitrary formula of size $n = |\mathrm{sub}(\varphi)|$, having modal depth $m = \mathrm{md}(\varphi)$. Suppose that $\mathfrak{M}, (r_h, r_v) \not\models \varphi$, for some delta product model $\mathfrak{M} = (\mathfrak{F}_h \times^\delta \mathfrak{F}_v, \mathfrak{V})$, such that $\mathfrak{F}_h = (W_h, R_h) \in \mathsf{Fr}\,\mathbf{K}$ and $\mathfrak{F}_v = (W_v, R_v) \in \mathsf{Fr}\,\mathbf{Alt}(t)$.

First, we define a sequence of subsets $V_0, \ldots V_m$ of W_v, by taking $V_0 = \{r_h\}$, and

$$V_{k+1} = \{y \in W_v : xR_vy \text{ for some } x \in V_k\},$$

for all $k > 0$. Note that since $\mathfrak{F}_v$ need not be a tree, these sets may or may not be disjoint. In either case, we necessarily have that $|V_{k+1}| \leq t \cdot |V_k|$ since no element of $\mathfrak{F}_v$ may have more than t distinct successors. It follows that $|V_k| \leq t^k$, for each $k \leq m$.

We define $\mathfrak{F}'_v = (W'_h, R'_v)$ to be the *finite* subframe of $\mathfrak{F}_v$ given by,

$$W'_v = \bigcup_{k=0}^{m} V_k \qquad \text{and} \qquad R'_v = R_v \cap (W'_v \times W'_v).$$

Note that $\mathfrak{F}'_v$ is not necessarily a *generated* subframe of $\mathfrak{F}_v$, however, since $\mathsf{Fr}\,\mathbf{Alt}(t)$ is closed under arbitrary subframes, we are assured that $\mathfrak{F}'_v$ is a frame for $\mathbf{Alt}(t)$. Moreover, it is readily noted that $|W_v'| \leq 1 + t + \cdots + t^m \leq (m+1) \cdot t^m$ is at most exponential in the size of φ, and is even linear for $t = 1$.

Next, for all $x \in W_h$ and $y \in W_v$, we define the type

$$\boldsymbol{t}(x, y) = \{\psi \in \mathrm{sub}(\varphi) : \mathfrak{M}, (x, y) \models \psi\}$$

to be the set of all subformulas of φ that are satisfied at $(x, y) \in W_h \times W_v$.

We define a sequence of subsets $U_0, U_1, \ldots, U_m$ of W_h, by taking $U_0 = \{r_h\}$, and defining U_{k+1} inductively as follows. Suppose we have already defined $U_k \subseteq W_h$, for some $k < m$, and suppose that $x \in U_k$, $y \in W_v'$ and $\Diamond_h \alpha \in \boldsymbol{t}(x, y)$. Then, by definition, $\mathfrak{M}, (x, y) \models \Diamond_h \alpha$. Hence, we may fix some $z = z_{(x,y,\alpha)} \in W_h$, such that $x R_h z$ and $\mathfrak{M}, (z, y) \models \alpha$. We define $U_{k+1} \subseteq W_h$, by taking

$$U_{k+1} = \{z_{(x,y,\alpha)} \in W_h : x \in U_k,\ y \in W_v' \text{ and } \Diamond_h \alpha \in \boldsymbol{t}(x, y)\}.$$

Again, we note that since $\mathfrak{F}_h$ need not be a tree, these sets may or may not be disjoint. However, we necessarily have that

$$|U_{k+1}| \leq |U_k| \cdot |W_v'| \cdot |\mathrm{sub}(\varphi)| \leq |U_k| \cdot \big((m+1) \cdot t^m\big) \cdot n$$

from which it follows that

$$|U_k| \leq \big(n \cdot (m+1) \cdot t^m\big)^k,$$

for all $k \leq m$.

We define $\mathfrak{F}_h' = (W_h', R_h')$ to be the *finite* subframe of $\mathfrak{F}_h$ given by,

$$W_h' = \bigcup_{k=0}^{m} U_k \qquad \text{and} \qquad R_h' = \bigcup_{k<m} R_h \cap (U_k \times U_{k+1}).$$

Here, we note that $|W_h'| \leq \sum_{k=0}^{m} |U_k| \leq (m+1)(n \cdot (m+1) \cdot t^m)^m$ is also, at most, exponential in the size of φ.

We then define a new model $\mathfrak{M}' = (\mathfrak{F}_h' \times^\delta \mathfrak{F}_v', \mathfrak{V}')$, by taking $\mathfrak{V}'(p) = \mathfrak{V}(p) \cap (W_h' \times W_v')$, for all propositional variables $p \in \mathrm{sub}(\varphi)$.

A straightforward induction on $k_h, k_v \leq \mathrm{md}(\varphi)$ reveals that, for all subformulas $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}, (x, y) \models \psi \qquad \Longleftrightarrow \qquad \mathfrak{M}', (x, y) \models \psi, \tag{I.H.}$$

whenever $x \in U_{m-k_h}$, $y \in V_{m-k_v}$, and $\mathrm{md}(\psi) \leq k_h, k_v$.

In particular, we have that $\mathfrak{M}', (r_h, r_v) \not\models \varphi$, since $r_h \in U_0$ and $r_v \in V_0$, where $\mathfrak{M}'$ is a product model for $\mathbf{K} \times^\delta \mathbf{Alt}(t)$ of size at most exponential in the size of φ. This is to say that $\mathbf{K} \times^\delta \mathbf{Alt}(t)$ has the exponential product fmp, as required. □

This effective bound on the size of the models for $\mathbf{K} \times^\delta \mathbf{Alt}(t)$, provides us with a decision procedure for determining its satisfiability problem; given a formula φ, we may non-deterministically 'guess' a delta product model $\mathfrak{M}$ of size at most exponential in the size of φ, and verify, in time polynomial in the size of $\mathfrak{M}$, whether φ is satisfiable in $\mathfrak{M}$.

Theorem 10.18. *The decision problem for* $\mathbf{K} \times^\delta \mathbf{Alt}(t)$ *is decidable in* coNExpTime.

It is known that the decision problem for $\mathbf{K} \times \mathbf{Alt}$ is even decidable in ExpTime (see Theorem 6.6 of [38]). While its exact complexity remains a mystery, this is still a notable improvement over the coNExpTime upper bound for $\mathbf{K} \times^\delta \mathbf{Alt}$ provided here. It is unclear whether the same techniques can be applied to secure a similar ExpTime upper bound for $\mathbf{K} \times^\delta \mathbf{Alt}$.

Question 10.19. What is the complexity of the decision problem for $\mathbf{K} \times^\delta \mathbf{Alt}$?

10.6 Discussion

The addition of a diagonal constant provides a natural way of connecting the horizontal and vertical dimensions of product logics, however, occasionally at the expense of vastly increasing the computational complexity of their decision problems.

In this chapter we have shown that, in contrast to the modest complexity of both $\mathbf{K} \times \mathbf{S5}$ and $\mathbf{K4.3} \times \mathbf{S5}$, whose decision problems may be decided in coNExpTime and 2ExpTime, respectively [83, 101], the decision problems for $\mathbf{K} \times^{\delta} \mathbf{S5}$ and $\mathbf{K4.3} \times \mathbf{S5}$ are both undecidable — albeit recursively enumerable.

Note, however, that the above theorems do not apply in cases where $\mathcal{C}_h$ comprises some class of transitive, but not necessarily weakly-connected, frames. In particular, it remains open whether $\mathbf{K4} \times^{\delta} \mathbf{S5}$ is undecidable? In [49], the authors provide a *'liftable'* many-one reduction from $\mathbf{K}$ to $\mathbf{K4}$ — both PSpace-complete [78] — which yields a many-one reduction from $\mathbf{K} \times L$ to $\mathbf{K4} \times L$, whenever L is Kripke complete. However, their reduction relies heavily on $\mathbf{K} \times L$ being characterised by those product frames having an intransitive tree as their horizontal component. This fails, in general, for $\mathbf{K} \times^{\delta} L$, as evidenced by Lemma 10.13, which exhibits some $\mathbf{K} \times^{\delta} \mathbf{K}$-satisfiable formula that cannot be satisfied in any delta product model whose horizontal component is cycle-free[†] of arbitrary length in the horizontal component.

Question 10.20. Is the decision problem for $\mathbf{K4} \times^{\delta} \mathbf{S5}$ decidable?

Taking $\mathcal{T}$ to be the class of all intransitive trees, it is straightforward to show that $\mathbf{K} \times \mathbf{K} = \mathsf{Log}(\mathcal{T} \times \mathcal{T})$, since by a straightforward unravelling argument, every rooted frame is the p-morphic image of an intransitive tree (see, for example [14]). However, the same is not true of delta products. Indeed, $\mathbf{K} \times^{\delta} \mathbf{K}$ is properly subsumed by $\mathsf{Log}(\mathcal{T} \times^{\delta} \mathcal{T})$.

Furthermore, since the decision problem for $\mathbf{K} \times \mathbf{K}$ is non-elementary [49], so too must be the decision problem for $\mathsf{Log}(\mathcal{T} \times \mathcal{T})$. However, falling outside the scope of Theorem 10.7, it remains open whether $\mathsf{Log}(\mathcal{T} \times^{\delta} \mathcal{T})$ is even decidable?

Question 10.21. Is the decision problem for $\mathsf{Log}(\mathcal{T} \times^{\delta} \mathcal{T})$ decidable?

The Gödel class with identify is a classic example of a undecidable fragment of first-order, whose equality-free fragment is known to be decidable; indeed, this observation

[†]By cycle, we mean an *undirected* path $(a_0, a_1, \ldots, a_n)$ such that $a_n = a_0$ and $a_k R a_{k+1}$ or $a_{k+1} R a_k$, for all $k < n$.

went undetected by Gödel, who mistakenly claimed that the methods employed in his proof for the decidability of the equality-free fragment could be extended to the full class, with equality [45, 48].

In [40], the authors introduce the *simple square fragment* $\mathcal{SF}$ as the decidable fragment of first-order logic, whose formulas are constructed from atomic binary predicates $P_i(x, y)$, by freely applying Boolean connectives and *relativised quantifiers* of the form

$$\exists z\big(R(x,z) \land \varphi(z,y)\big) \qquad \text{and} \qquad \exists z\big(R(y,z) \land \varphi(x,z)\big),$$

where R is a built-in binary predicate. Unlike the decidable two-variable fragment of first-order logic, $\mathcal{SF}$ is situated within the undecidable three-variable fragment [122, 123]. This simple square fragment is readily seen to be the image of $\mathcal{ML}_2$ under the following variation of the standard translation $(\cdot)^* : \mathcal{ML}_2 \to \mathcal{L}$,

$$p_j^* = P_j(x,y), \quad \text{for } p_i \in \textsc{prop}, \qquad (\neg\psi)^* = \neg\psi^*, \qquad (\psi_1 \land \psi_2)^* = \psi_1^* \land \psi_2^*,$$
$$(\Diamond_h\psi)^* = \exists z\big(R(x,z) \land \psi^*(z/x,y)\big), \qquad (\Diamond_v\psi)^* = \exists z\big(R(y,z) \land \psi^*(x,z/y)\big),$$

where $R, P_0, P_1, \ldots$ are binary predicate symbols. Moreover, it is straightforward to check that, for all $\varphi \in \mathcal{ML}_2$, we have that $\varphi \in \mathbf{K} \times \mathbf{K}$ if and only if φ^* is a theorem of first-order logic. Since the decision problem for $\mathbf{K} \times \mathbf{K}$ is decidable, so too must be the simple square fragment $\mathcal{SF}$ [40].

However, by expanding $\mathcal{SF}$ to include equality, the fragment $\mathcal{SF}_\approx$ is readily seen to be the image of $\mathcal{ML}_2^\delta$ under the following extension of $(\cdot)^*$ that interprets $\delta^* = (x \approx y)$. It is similarly straightforward to check that, for all $\varphi \in \mathcal{ML}_2^\delta$, we have that $\varphi \in \mathbf{K} \times^\delta \mathbf{K}$ if and only if φ^* is a theorem of first-order logic with equality. Hence, it follows from Theorem 10.7 that the simple square fragment *with equality* is undecidable.

Thus $\mathcal{SF}_\approx$ represents yet another example of an undecidable fragment of first-order logic, whose equality-free fragment is decidable.

Appendix A

Complexity Theory

In this section we provide a brief overview of some standard results and terminology from complexity and recursion theory that will be understood implicitly throughout this thesis. For a more detailed discussion, see, for example [43, 90, 19, 109].

A.1 Complexity Theory

A *decision problem*, is a question to which only the answers "yes" and "no" are expected. This can be better formalised as a membership problem consisting of a given *universe* U, equipped with a measure $||\cdot|| : U \to \omega$, and a subset $X \subseteq U$ of *affirmatives* [43].

In the context of modal logic, the universe is taken to be the set of modal formulas $\mathcal{ML}_n$, while X is taken to be the particular modal logic in question; the measure $||\cdot|| : \mathcal{ML}_n \to \omega$ is typically taken to be the *size* of the formula.

A *Turing machine* is an idealised machine that operates on a two-way infinite tape, divided into discrete units. The machine reads from and writes to the tape with symbols from a fixed finite alphabet A, in accordance to a given set of instructions. The machine is said to be *deterministic* if its operation is uniquely determined by its instructions, and *non-deterministic* otherwise. A more detailed account can be found in [19].

A Turing machine may take any finite string of letters from A as an input, written to the tape, and may *accept* or *reject* the string upon terminating, in accordance to its instructions. It is possible that a machine may neither accept nor reject upon certain input strings, and instead fail to terminate entirely. The contents of the tape upon termination may be interpreted as the value of a partial function computed by the Turing machine for a given input. Any partial function that can be computed by a Turing machine is said to be *recursive*. We may characterise the relative complexity of various decision problems in terms of the temporal and spatial demands that any Turing machine faces in answering their membership problem.

A decision problem $X \subseteq U$ is said to be *decidable* if there is some Turing machine that terminates on every input $u \in U$ (appropriately encoded), and accepts if and only if $u \in X$ We say that X is *undecidable* should no such machine exist.

In finer granularity, for each monotonic function[†] $f : \omega \to \omega$, we define the complexity classes $\textsc{Time}(f(n))$, $\mathrm{N}\textsc{Time}(f(n))$ and $\textsc{Space}(f(n))$ as follows. For every decision problem $X \subseteq U$, we have that $X \in \textsc{Time}(f(n))$ (resp. $X \in \mathrm{N}\textsc{Time}(f(n))$) if there is some $c < \omega$ and some deterministic (resp. non-deterministic) Turing machine that terminates on every input $u \in U$ (appropriately encoded), within a number of steps bounded by $c \cdot f(||u||)$, and accepts if and only if $u \in X$.

We say that $X \in \textsc{Space}(f(n))$ if there is some $c < \omega$ and some deterministic Turing machine that terminates on every input $u \in U$, having used no more than $c \cdot f(||u||)$ units of tape, and accepts if and only if $u \in X$.

We define the following complexity classes, each closed under polynomial reductions:

$$\mathrm{P} := \bigcup_{m<\omega} \textsc{Time}(n^m), \qquad \mathrm{NP} := \bigcup_{m<\omega} \mathrm{N}\textsc{Time}(n^m), \qquad \mathrm{P}\textsc{Space} := \bigcup_{m<\omega} \textsc{Space}(n^m).$$

For each $0 < k < \omega$, we define $\exp_k : \omega \to \omega$ inductively, by taking

$$\exp_1(n) = 2^n \qquad \text{and} \qquad \exp_{k+1}(n) = 2^{\exp_k(n)}.$$

We then define the following complexity classes, for each $0 < k < \omega$:

$$\begin{aligned} k\textsc{ExpTime} &:= \bigcup_{m<\omega} \textsc{Time}(\exp_k(n^m)) \\ \mathrm{N}k\textsc{ExpTime} &:= \bigcup_{m<\omega} \mathrm{N}\textsc{Time}(\exp_k(n^m)) \\ k\textsc{ExpSpace} &:= \bigcup_{m<\omega} \textsc{Space}(\exp_k(n^m)) \\ \textsc{Elementary} &:= \bigcup_{k<\omega} k\textsc{ExpTime} \end{aligned}$$

[†]A function $f : \omega \to \omega$ is said to be *monotonic* if $f(n) \leq f(m)$, whenever $n \leq m$.

Where $k = 1$, we simply write ExpTime, NExpTime, and ExpSpace, respectively. The class PrimRec extends Elementary, and comprises all those decision problems that admit a decision procedure whose times is bounded by some *primitive recursive* function.

The *Ackermann function* [4, 95] $A : \omega \times \omega \to \omega$ is given by,

$$A(n,m) = \begin{cases} m+1 & \text{if } n = 0, \\ A(n-1,1) & \text{if } n > 0 \text{ and } m = 0, \\ A\big(n-1, A(n,m-1)\big) & \text{if } n > 0 \text{ and } m > 0, \end{cases}$$

for all $n, m < \omega$, and is among the first examples of a non-primitive recursive function that is, nonetheless, computable in principle. We denote by Ackermann, the class of all decision problems that admit a decision procedure whose time is bounded by some primitive recursive function of $A(||x||, ||x||)$. It should be clear that Ackermann strictly subsumes the class PrimRec.

The class Rec comprises every decision problem that admits a procedure whose time is bounded by any recursive function; it is posited that this encapsulates every decidable decision problem.

Together, these classes form the following nested hierarchy:

$$\begin{aligned} \mathrm{P} \subseteq \mathrm{NP} \subseteq \textsc{PSpace} \subseteq \textsc{ExpTime} \subseteq \textsc{NExpTime} \subseteq \textsc{ExpSpace} \\ \subseteq 2\textsc{ExpTime} \subseteq \mathrm{N}2\textsc{ExpTime} \subseteq 2\textsc{ExpSpace} \cdots \subseteq \textsc{Elementary} \\ \subseteq \textsc{PrimRec} \subseteq \textsc{Ackermann} \subseteq \textsc{Rec}. \end{aligned}$$

Additionally, for each complexity class $\mathscr{C}$, we define the class co$\mathscr{C}$ comprising all those decision problems $X \subseteq U$ whose *complement* $(U - X) \subseteq U$ belong to $\mathscr{C}$. Note that $\mathscr{C} = \mathrm{co}\mathscr{C}$, whenever $\mathscr{C}$ is a deterministic class, however it remains open whether the same is true of non-deterministic classes.

A *many-one reduction* between two decision problems, $X \subseteq U_1$ and $Y \subseteq U_2$, is a recursive function $f : U_1 \to U_2$, computable by some Turing machine T, such that

$$x \in X \qquad \Longleftrightarrow \qquad f(x) \in Y.$$

In which case, we say that X is *many-one reducible* to Y. More specifically, we say that f is a *polynomial time reduction* if there is some *polynomial function* $p : \omega \to \omega$ such that T terminates on every input $x \in X$ within a number of steps bounded by $c \cdot p(||x||)$. In which case, we say that X is *(polynomially) reducible* to Y.

A problem X is said to be $\mathscr{C}$*-hard* if every decision problem $Y \in \mathscr{C}$ is polynomially reducible to X, and $\mathscr{C}$*-complete* whenever X is both $\mathscr{C}$-hard and decidable in $\mathscr{C}$.

A.2 Recursion Theory

While this distinction between decidable and undecidable problems is an important dichotomy to make, we can employ the tools of recursion theory to gain a better understanding of the relative complexity of undecidable problems.

We say that a set of natural numbers $X \in 2^\omega$ is *defined* by the formula $\varphi(x)$ of *Peano arithmetic* if, for all $m < \omega$,

$$m \in X \qquad \Longleftrightarrow \qquad (\omega, +, 0) \models \varphi(m).$$

That is to say $\varphi(x)$ is satisfiable in the standard model of Peano arithmetic under the variable assignment $h(x) = m$. We say that $X \in 2^\omega$ belongs to the class Σ^0_n if it is defined by some formula

$$\varphi(x) = \exists y_1 \forall y_2 \ldots \forall y_n (\exists y_n)\ \psi(x, y_1, \ldots, y_n)$$

of *Peano arithmetic*, such that ψ contains no unbounded quantifiers. Formulas of the above syntactic variation, and those semantically equivalent to such formulas, are appropriately termed Σ^0_n-formulas.

For each $n < \omega$, we define the class Π^0_n of those sets whose complements belong to Σ^0_n, such that

$$X \in \Pi^0_n \qquad \Longleftrightarrow \qquad (\omega - X) \in \Sigma^0_n.$$

Appropriately, we say that a formula is a Π^0_n-formula if it is semantically equivalent to the negation of a Σ^0_n-formula.

Furthermore, for each $n < \omega$, we take $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$, to be the class of all sets of natural numbers definable by *both* a Σ_n^0-formula and a Π_1^0-formula.

$$
\begin{array}{ccccccccccccc}
 & \Sigma_1^0 & & & \Sigma_2^0 & & & \Sigma_3^0 & & & \Sigma_4^0 & \subseteq & \cdots \\
\rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & \rotatebox{45}{$\subseteq$} & & & \\
\Delta_1^0 & & & \Delta_2^0 & & & \Delta_3^0 & & & \Delta_4^0 & & & \\
\rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\supseteq$} & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\supseteq$} & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\supseteq$} & \rotatebox{-45}{$\subseteq$} & & & \\
 & \Pi_1^0 & & & \Pi_2^0 & & & \Pi_3^0 & & & \Pi_4^0 & \subseteq & \cdots
\end{array}
$$

Figure A.1: The Arithmetic Hierarchy.

More complex still, are those sets of natural numbers definable only by *second-order arithmetic*. We say that a set $X \in 2^\omega$ belongs to the class Σ_1^1 if it is defined by some formula:

$$\exists P_1 \ldots \exists P_n \ \psi(x)$$

of second-order arithmetic, such that ψ contains no second-order quantifiers. Such sets are said to be *analytic*, while those not definable by such a formula are said to be *non-analytic*.

We define the class Π_1^1 of those sets whose complements belong to Σ_1^1. These two classes constitute the base of what is termed the *analytic hierarchy*, which can be continued in a similar fashion to that of the arithmetic hierarchy, with the prefix of second-order quantifiers alternating accordingly. However, such extensions go beyond the scope of this thesis.

A problem X is said to be *$\mathscr{C}$-hard* if every decision problem $Y \in \mathscr{C}$ is many-one reducible to X, and *$\mathscr{C}$-complete* whenever X is both $\mathscr{C}$-hard and decidable in $\mathscr{C}$, where $\mathcal{C}$ is some *undecidable* class of decision problems.

Appendix B

ω-Reachability for Counter Machines

In this appendix we show that the ω-REACHABILITY problem for incrementing counter machines is Π^0_1-complete — that is to say, undecidable, yet co-recursively enumerable — matching that of the BÜCHI problem.

This is somewhat surprising, given that, in the case with lossy counter machines, the complexities of the ω-REACHABILITY and the BÜCHI problems lie on opposite side of the arithmetic hierarchy; with the ω-REACHABILITY problem being Π^0_1-complete, while the BÜCHI problem is Σ^0_1-complete.

It was proved in [107] that the UNBOUNDEDNESS problem for lossy counter machines is Π^0_1-hard. Here, we will reduce this problem to that of the ω-REACHABILITY problem for incrementing counter machines, thereby providing us our undecidable lower bound.

LCM UNBOUNDEDNESS: (Π^0_1-complete [107])

Given an incrementing counter machine $\mathcal{M}$, are infinitely many configurations reachable from $(q_{\mathsf{init}}, \vec{0})$?

Theorem B.1. *The* ω-REACHABILITY *problem for incrementing counter machines is* Π^0_1*-hard.*

Proof. We show that the UNBOUNDEDNESS problem for lossy counter machines is reducible to the ω-REACHABILITY problem for incrementing counter machines.

Let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary lossy counter machine, and recall the definition of $\mathcal{M}^{\mathsf{op}}$ given in Section 9.3.2. We augment $\mathcal{M}^{\mathsf{op}}$ by introducing two additional counter t_0, t_1 as well as the following additional control states and transitions, depicted below in Figure B.1.

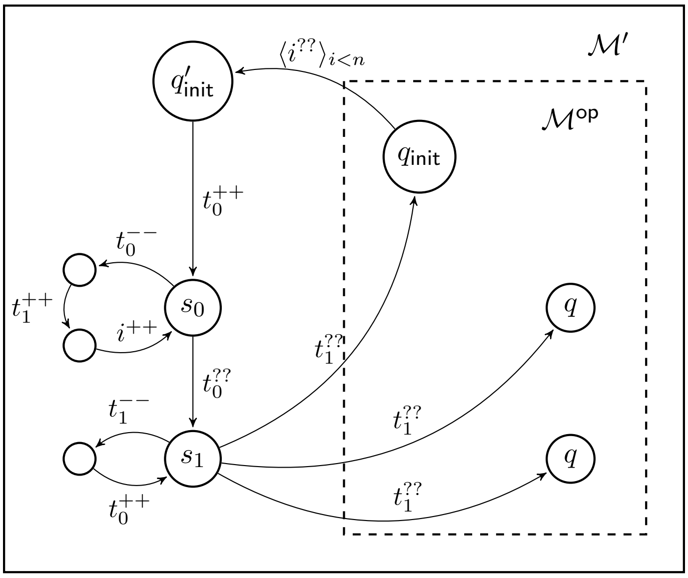


Figure B.1: Illustration of the counter machine $\mathcal{M}'$.

A sequence of n transitions lead from $q_{\mathsf{init}} \in Q$ to the new initial state $q'_{\mathsf{init}} \in Q'$, testing the emptiness of each of the counters $i < n$. Transitioning from q'_{init} to s_0, the machine increments counter t_0.

At s_0 the contents of counter t_0 are copied into the counters t_1 and, non-deterministically distributed over the counters $i < n$, emptying t_0 in the process. Transitioning to s_1, the contents of counter t_0 are restored by copying back the contents of t_1. The machine then, non-deterministically transitions to some control state of $\mathcal{M}^{\mathsf{op}}$, returning to q'_{init} only via $q_{\mathsf{init}} \in Q$, which requires all counters $i < n$ to be empty.

We show that the following statements are equivalent:

(i) $\mathcal{M}'$ has a computation that visits $q'_{\mathsf{init}} \in Q'$ infinitely often.

(ii) $\mathcal{M}'$ has a computation that visits $q'_{\mathsf{init}} \in Q'$ at least n times,

(iii) $\mathcal{M}$ is unbounded

(i) ⇒ (ii) Trivial.

(ii) ⇒ (iii) Suppose that, for each $k < \omega$, there is a computation that visits q'_{init} at least k-times. During the kth cycle, the value of t_0 is at least k, which is distributed to the counters of C at s_0. To return to q'_{init} there must be some $(q, v) \in \mathrm{Conf}_{\mathcal{M}}$ such that $\sum_{i<n} v(i) \geq k$, and $(q_{\mathsf{init}}, \vec{0})$ is $\mathcal{M}^{\mathsf{op}}$-reachable from (q, v) by some incrementing computation Consequently, for each $k < \omega$ there is some configuration $(q, v) \in \mathrm{Conf}_{\mathcal{M}}$ such that $\sum_{i<n} v(i) \geq k$ and (q, v) is $\mathcal{M}$-reachable from $(q_{\mathsf{init}}, \vec{0})$ by some *lossy* computation. This is to say that $\mathcal{M}$ is unbounded.

(iii) $\Rightarrow$ (i) Suppose that $\mathcal{M}$ is unbounded. Then for every $k < \omega$ there is some $(q_k, v_k) \in \mathrm{Conf}_{\mathcal{M}}$ such that $\sum_{i<n} v_k(i) \geq k$ and (q_k, v_k) is $\mathcal{M}$-reachable from $(q_{\mathsf{init}}, \vec{0})$ by some lossy computation. Consequently $(q_{\mathsf{init}}, \vec{0})$ is $\mathcal{M}^{\mathsf{op}}$-reachable from (q_k, v_k) by some incrementing computation.

We construct a non-terminating incrementing computation that visits $q'_{\mathsf{init}} \in Q'$ infinitely often. Firstly, there is trivially an incrementing computation that visits q'_{init} at least once, since q'_{init} is the initial state of $\mathcal{M}'$.

Now suppose that r_k is an partial incrementing computation that visits q'_{init} exactly k times, terminating at $(q'_{\mathsf{init}}, \vec{0}, n, 0)$. We extend r_k to a computation r_{k+1} by transitioning to $(s_0, \vec{0}, k+1, 0)$, whereupon the contents of t_0 are distributed among the first n counters, incurring any necessary incrementing errors to reach the configuration $(s_0, v_{k+1}, 0, k+1)$.

Transitioning to $(s_1, v_{k+1}, 0, k+1)$, we move the contents of counter t_1 back into t_0 before non-deterministically transitioning to $(q_{k+1}, v_{k+1}, k+1, 0)$.

From there, we simulate the computation of $\mathcal{M}^{\mathsf{op}}$, terminating at $(q_{\mathsf{init}}, \vec{0}, k+1, 0)$ before returning to $(q'_{\mathsf{init}}, \vec{0}, k+1, 0)$, as required.

By induction we may construct a non-terminating incrementing computation that visits $q'_{\mathsf{init}} \in Q$, infinitely often.

Since the Unboundedness problem for lossy counter machines is Π^0_1-hard, so too must be the ω-Reachability problem for *incrementing* counter machines, as required. $\square$

Note that, not only does the above proof demonstrate the Π^0_1-hardness for the ω-Reachability problem for incrementing counter machines, it also provides an alternative proof for the Π^0_1-hardness of the Büchi problem for incrementing counter machines. Moreover, this prove differs from the one presented by [29, 93], which employed an elaborate reduction from non-Termination problem for reliable counter machines.

Theorem B.2. *The* ω-REACHABILITY *problem for incrementing counter machines belongs to the class* Π^0_1.

Proof. Let $\mathcal{M} = (Q, q_{\mathsf{init}}, n, \Delta, H)$ be an arbitrary incrementing counter machine, and suppose that $\ell \in Q$ is a given control state. We may assign a 'toll charge' to $\mathcal{M}$ by introducing a new counter machine t, which is decremented upon exiting the control state ℓ, and introducing a new control state q_{fin}, which can only be accessed upon emptying the contents of counter t. Upon entering the control state q_{fin}, we may empty the contents of all remaining counters. Formally, we define $\mathcal{M}^{\mathsf{toll}} = (Q^{\mathsf{toll}}, q_{\mathsf{init}}, (n+1), \Delta^{\mathsf{toll}}, H)$, by taking

$$Q^{\mathsf{toll}} \;=\; Q \cup \{q_{\mathsf{fin}}\} \cup \{s_{(\alpha,q)} : (\ell, \alpha, q) \in \Delta\},$$

and

$$\begin{aligned} \Delta^{\mathsf{toll}} \;=\; & \{(\ell, n^{--}, s_{(\alpha,q)}), (s_{(\alpha,q)}, \alpha, q) : (\ell, \alpha, q) \in \Delta\} \\ & \cup \{(q_{\mathsf{fin}}, i^{--}, q_{\mathsf{fin}}) : i < n\} \cup \{(q_{\mathsf{fin}}, n^{--}, q_{\mathsf{fin}})\}. \end{aligned}$$

It follows that $\mathcal{M}$ has an incrementing computation that visits $\ell \in Q$ at least k times if and only if $(q_{\mathsf{fin}}, \vec{0}, 0)$ is $\mathcal{M}^{\mathsf{toll}}$-reachable from $(q_{\mathsf{init}}, \vec{0}, k)$.

Now, suppose that $f : \omega \to \mathrm{Conf}_{\mathcal{M}^{\mathsf{toll}}}$ is some fixed enumeration of the configuration space of $\mathcal{M}^{\mathsf{toll}}$. Since there REACHABILITY problem for incrementing counter machines is decidable — albeit, non-primitive recursive — there is some recursive predicate $M \subseteq \omega \times \omega$ such that

$$M(x, y) \qquad \Longleftrightarrow \qquad f(y) \text{ is } \mathcal{M}^{\mathsf{toll}}\text{-reachable from } f(x).$$

Moreover, let $S \subseteq \omega \times \omega$ and $T \subseteq \omega$ be two recursive predicates, defined such that

$$\begin{aligned} S(x, k) \;&\Longleftrightarrow\; f(x) = (q_{\mathsf{init}}, \vec{0}, k), \\ T(x) \;&\Longleftrightarrow\; f(x) = (q_{\mathsf{fin}}, \vec{0}, 0). \end{aligned}$$

It follows that $\mathcal{M}$ has a computation that visits $\ell \in Q$ at least k times, for each $k < \omega$ if and only if

$$\forall k \forall x \forall y \; \big(S(x, k) \wedge T(y) \to M(x, y)\big)$$

is a theorem of Peano arithmetic. Hence it follows that the ω-REACHABILITY problem for incrementing counter machines belongs to the class Π^0_1, as required. $\square$

Hence it follows that the ω-Reachability problem for incrementing counter machines is Π^0_1-complete, akin to that of the ω-Reachability problem for *lossy* counter machines. It is worth noting that, although the exact computational complexity of the ω-Reachability problem for *reliable* counter machines remains open, a small modification to the above proof does yield an Π^0_2 upper bound.

Corollary B.3. *The ω-Reachability problem for reliable counter machines belongs to the class Π^0_2.*

Proof. The proof is identical to that of Theorem B.2, except in the detail that the Reachability problem for reliable counter machines is Σ^0_1-complete. Thus by definition, there is some Σ^0_1-formula $\exists z\ M(x, y, z)$ such that

$$\exists z\ M(x, y, z) \qquad \Longleftrightarrow \qquad f(y) \text{ is } \mathcal{M}^{\mathsf{toll}}\text{-reachable from } f(x).$$

It follows that $\mathcal{M}$ has a computation that visits $\ell \in Q$ at least n times, for each $n < \omega$ if and only if

$$\forall k \forall x \forall y \exists z\ \big(S(x, k) \wedge T(y) \to M(x, y, z)\big)$$

is a theorem of Peano arithmetic. Hence it follows that the ω-Reachability problem for reliable counter machines belongs to the class Π^0_2, as required. $\square$

This is in notable contrast with the lofty Σ^1_1-completeness of the Büchi problem for reliable counter machines.

Question B.4. What is the complexity of the ω-Reachability problem for reliable counter machines?

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